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RESEARCH ARTICLE

# To Tell a Story about Students' Exploring on Mathematical Objects based on Groups Cognitive Style: Commognitive Perspective

Rita Lefrida [1*], Bakri M[2], Sutji Rochaminah[3], Mubarik[4]

[1]Tadulako University, Indonesia

[2,3,4] Tadulako University, Indonesia,

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***Corresponding Author:**

lefrida@yahoo.com

**ABSTRACT**

The qualitative approach in this research is exploratory. It was conducted at Universitas Tadulako, a university located in Palu, Indonesia. The study aims to describe how students construct mathematical objects through student participation in derivative discourse in a commognitive perspective. This is based on the theory that mathematics is a discourse that creates its own object. In this study, the subjects were recruited and divided based on their cognitive style. There were six subjects: three subjects in the cognitive style field dependent group and three subjects in the cognitive style field independent group. Furthermore, the subjects' ways of constructing mathematical objects into signifier and realization forms were the focus of the research. The results of the analysis showed that the cognitive style field dependent group utilized more trial-and-error and were able to relate the problems their previous experiences. On the other hand, the cognitive style field independent group was more able to overcome the problem by getting a relationship between signifier and realization.

## INTRODUCTION

The words communication and cognition are formed by the term commognition which is known as the commognitive framework (Sfard, 2008; 2020. Based on the commognitive perspective, thinking is a form of communication (interpersonal) that is individualistic, which emphasizes that the process of thinking and communicating is unity. Accordingly, when doing mathematical tasks, a person will communicate mathematically with himself which can be called thinking Lavie et al. (2019). Commognition is fundamentally a theory of participation as learning only occurs through the participation of individual thinking in mathematical discourse (Berger, 2013; Kim, Choi & Lim, 2017).

According to the commognition framework, learning mathematics is defined as the process of changing and expanding students' discourse that can be seen when they are communicating (Ben-Zvi & Sfard, 2007; Kieran, 2001; Sfard, 2007, 2008; Tabach & Nachlieli, 2016). Furthermore "If learning means change, what is it that changes when people learn?, the content for capturing human uniqueness" (Sfard, 2008; 2017). A mathematician Mark Kac added "I cannot define this creature, but I recognize it when I see it (Sfard, 2008)." Thus, learning is not only knowing but seeing what is being done when doing it

Activities that are closely related in commognition are those related to objects and that consider the subject as the actor of discursive actions. Learning is a commognitive activity, which includes reasoning, abstracting, objectifying, and subjectifying. In a classroom setting, the four activities above are rarely shown in clear form because they are embedded in different types of communication (Sfard, 2008). The first type is mathematizing, and it happens when students explain about mathematical objects. Another thing that often occurs is subjectifying, or in other words communicating about mathematizers. One type of subjectifying that students use to direct their mathematical steps is meta-mathematizing (Chan & Sfard, 2020).

Accordingly, mathematics is a multilayered recursive structure of discourses about discourse (Sfard, 2008). Mathematics is an autopoietic system, which is a system that creates its own objects. Furthermore, it is also a system that produces the things talked about. Mathematics is also a multi-layered recursive structure of discourse-about-discourse. The objects are explored when thinking or through collaboration. Furthermore, this object develops through communication which is the purpose of mathematics. Therefore, there is a reason for saying mathematics is a discourse because mathematics is a certain form of communication that is well defined (Sfard, 2007; 2008; 2012; 2017). Mathematics is a discourse in which we tell stories about mathematical objects. Furthermore, when in a discourse, one can talk about the mathematical object of the discourse (Karavi., et al. 2022).

Mathematical objects are in the form of facts, concepts, operations and principles (Soejadi, 2010). The facts are in the form of conventions or certain symbols. Concepts are closely related to definitions. Operations can be viewed as a function. Principles can be axioms, theorems, and properties. According to the perspective of commognition, mathematical objects are constructed through discourse, such as numbers, functions, sets, and geometric shapes (Sfard, 2008). Mathematical objects are divided into concrete and abstract (Baccaglini-Frank, 2021). Mathematical objects are abstract discursive objects with clear mathematical signifiers, namely the relationship between signifier and realization. Several studies have explored on mathematics objects, namely Roberts dan Roux (2018) on linear equations and Nachlieli dan Tabach (2012) on functions. Sfard suggested that, when students solve problems such as derivatives, they would begin with the signifier in the form of symbols or words. After that, the signifier would result in a response (written or verbal), which is realization. In the present study, mathematics objects in derivatives are discussed.

Furthermore, "the different types of communication, and thus of commognition, that draw some individuals together while excluding some others will be called discourses", which can be interpreted as a type of communication that involves several people and at the same time excludes several others called discourse (Sfard, 2008). Accordingly, that mathematical discourse is characterized by keywords, visual mediators, endorsed narrative, and routines (Ben-Zvi & Sfard , 2007; Sfard, 2008; Nardi et al., 2014).

From the point of view of commognition, in mathematics courses at higher education, a lecturer and students participate in mathematical discourse. Their participation is manifested in the use of keywords with characteristics (e.g., "limit," "continuous","differentiable"). Visual mediators are visible objects that serve to communicate relationships and operations with mathematical objects. Some examples of visual mediators are concrete "$f'(x)$" symbols, diagrams, tables, graphs, and symbols of formal notation used in algebra. A narrative is called an endorsed narrative "if it can be derived according to generally accepted rules from other endorsed narratives". Endorsed narratives are often labeled as truth, a set of propositions that are accepted and labeled as true by a particular community. In the case of mathematical discourse, the solicitation narrative must be constructed and proven according to a well-defined set of rules (mathematical theory). Learning is seen as routine because it is defined as a pair of task-procedure, "routine performed in a given task situation is a task, as seen by the performer, along with the procedures he or she carries out to perform the task" (Lavie et al. 2019).

The socio-constructivist approach that focuses on discourse in the student learning process can be said to be effective (Sfard & Cobb, 2014). The 21st century generation is required to be able to communicate and collaborate well, namely working with certain individuals and communities. According to Binkley (2012) there are four skills that must be possessed by the 21st century generation including ways of working (communication, collaboration, and teamwork). Furthermore, the flow of Kurikulum Merdeka that is echoed in Indonesia also focuses on communication and collaboration in problem solving.

In mathematics learning, collaborative settings in groups are proved useful (Kyndt et al., 2013; Springer et al., 1999; Heyd-Metzuyanim et al, 2023). Student-centered learning (central participant) in certain discourses is effective in describing the learning process (Lerman, 2000; Sfard, 2008; Sfard & Cobb, 2014). The increasing number of group discussions and collaborative assignments in the classroom benefits students because it helps them engage and build confidence (Rittle-Johnson et al., 2021). On the other hand, PISA 2015 released that Chinese students are not very suitable for collaborative problem-solving tasks because their teaching is accustomed to a traditional teacher-centered environment (Zhou & Lu, 2017; Lu, et al., 2022).

Research on cognitive processes (Cai & Gu, 2015; Liang et al., 2017) can be used as a basis for research on a commognitive perspective. This is because the more developed cognition will affect the development of commognition and the causes of cognitive style (Lefrida, 2021). Based on the opinions, commognition and cognitive style have an important role in learning, especially the process of solving derivative questions. Research that has adopted Sfard's theory to examine mathematical thinking are about functions (Nachilieli & Tabach, 2012; Tabach & Nachlieli, 2016), calculus (Park, 2015; Nardi et al., 2014; Ng, 2016, 2018; Lefrida et al, 2021; Wille, 2020), geometri (Wang & Kinzel, 2014); Heyd-Metzuyanim et al. (2023), linear equations (Robert & Roux, 2018), as well as mathematical modeling problems (Rogovchenko, 2022).

This study focuses on commogniton theory where the process of constructing mathematics objects in this case is about derivation that is carried out collaboratively in the cognitive style group. Based on the description above, with the differences in cognitive style, there is the possibility of student commognition in constructing different mathematics objects. Is this correct? Of course, research needs to be done to answer the question. Designing focused group work and communicating effectively are necessary for ongoing research in mathematics education.

## METHODOLOGY

### Sample and Data Collection

The type of research used is exploratory research with a qualitative approach. The research subjects were selected from students of the Mathematics Education Study Program, Universitas Tadulako in the 2022/2023 academic year. The selection of research subjects based on instrument Group Embedded Figure Test (GEFT). This instrument contains 25 items. The cognitive style data obtained were then analyzed using Microsoft excel program. The results were used to determine the cognitive style field dependent and field independent. The subjects in this study were six people, who were divided into 2 groups of 3 members each. This study used triangulation method with within-method type, because the researcher used one method, namely Focus Group Discussion and several within strategies, namely with 3 different derivative tasks, combining questions and answers according to the role of the researcher. Examination of data credibility of researchers can use data convergence criteria to the same meaning.

Furthermore, these two groups of subjects were given three derivative tasks as demonstrated in Table. 1. They work on tasks via discussion, through the autopoietic system which is a system that creates its own objects so that it can be seen how the subject constructs mathematical objects based on the theory of commognition.

**Table 1. Derivation Task**

| Task 1 |
|---|
| The left and right derivatives f of x is defined by<br>$f'_{-}(x) = \lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h}$ and $f'_{+}(x) = \lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h}$<br>if the limit is there. Next $f'(x)$ is present if and only if the left derivative and the right derivative are present and the value is the same.<br>a. Get $f'_{-}(1)$ and $f'_{+}(1)$ for function $f(x) = \begin{cases} -1-2x & \text{if} \quad x < -1 \\ x^2 & \text{if} \quad -1 \le x \le 1 \\ x & \text{if} \quad x > 1 \end{cases}$<br>b. Where is it f differentiable?<br>c. Is f continuous? |

d. Draw f(x) graphs and $f'(x)$ (Stewart, 2010)

Task 2

It is known

$$f(x)=\begin{cases} 0 & if \quad x\le 0 \\ 5-x & if \quad 0<x<4 \\ \dfrac{1}{5-x} & if \quad x\ge 4 \end{cases}$$

a. Determine $f'_{-}(4)$ and $f'_{+}(4)$ b. Chart sketch of $f(x)$

c. Where does $f$ discontinue? d. Where $f$ is undifferentiated? (Stewart, 2010)

Task 3

Some functions with information on formulas and their properties are given as follows:

a. $f(x) = |x|$, continuous, has no derivatives at $x = 0$.
b. $f(x) = [\![x]\!]$, not continuous, has no derivative at x = 0.
c. $f(x) = |x-1|$, continuous, has no derivatives at x = 1.
d. $f(x) = x^2$, continuous, has a derivative at x = 0.
e. $f(x) = x|x|$, continuous, has a derivative at x = 0.
f. $f(x) = x^2 - 1$, continuous, has a derivative at x = 0.
g. $f(x) = \frac{x}{|x|}$, not continuous, has no derivative at x = 0.

What are the characteristics of a function that have derivatives? (Lefrida et at., 2023)

Tasks 1 and 2 are taken from (Stewart, 2010) which are adapted to research needs. Then language validation was carried out by experts in the field of mathematics. Task 3 was taken from (Lefrida et al., 2023) which was used to explore students' mathematical objects.

**Analyzing of Data**

Data were collected through video recordings of subjects when they discussed in groups working on the tasks. The data obtained were analyzed to see how the subject constructs mathematics objects of derivations through discourse from a commognition perspective. The tasks were assigned to each cognitive style group and completed by them in discussion. All their conversations during the discussion were recorded and transcribed. The following description of student commognition in constructing mathematics objects was reviewed based on the cognitive style group.The research design was set as shown in Figure 1.

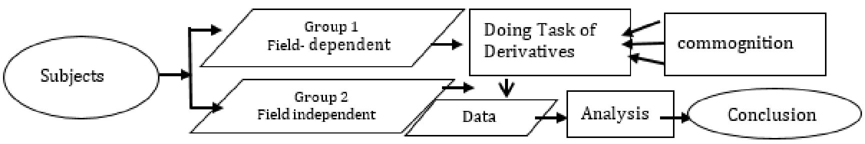



**Figure 1. Research Design**

The following is a description of student commognition in constructing mathematical objects based on cognitive style.

## FINDINGS/RESULTS

### Description of Constructing Mathematics Objects in Derivation based on the Cognitive Style Field Dependent (K1)

The subjects in this field dependent group consist of three students/subjects. The three subjects are given initial by NI, NU and PU. They were sitting face to face when they are doing the tasks.

### Task 1

### Determining the left derivative and the right derivative

In task I, researchers write the definition of the right derivative and the left derivative along with their symbol. Researchers asked the three subjects to read the questions in part a in turn with the aim of how the subjects recited the symbols and in accordance with what they wrote. The three subjects still misread the symbol $f'_{-}(1)$ and $f'_{+}(1)$. The (-) sign was read as "minus" and the (+) sign was read "plus". The discussion transcript is given in Table 2.

**Table 2. KITIa Discussion Transcripts**

| Interview | Discussion Transcript |
|---|---|
| Researcher | What can be captured or the essence of the problem a? |
| NU | We need to determine left and right limits |
| Researcher | Read the question again! |
| NU | It has a derivation if the left limit and the right limit exist |
| NI | The left limit and right limit values must be the same |
| NI | We define the functions to be used |
| | The subject begins to discuss question a)<br>Occasionally they look at each other |
| NU | For the right limit this function is used (pointing to the question) |
| NI and PU | Yes this one (while pointing too). |
| Resercher | Which one is used for the right derivative? |
| NU and PU | This one (pointing to the function on the problem) $x$ is greater than 1 |
| NI | How about the right one (also points to the function on the question) |
| | The three subjects determined the function and they wrote it |
| PU | Every step of the question he wrote |
| NU | Writing with occasional glances at NU paper |
| NI | Speaking while writing in a low voice. |
| NU | I got negative 2 |
| NI | Which ones? |
| NU | Derivation on the right |
| NU | Right derivation |
| PU | Yes. |
| NU | Notice zero from the right or from the left |
| | The left and right values are different. |
| PU | Not yet. |
| NU | I get -2. |
| PU | I can't answer yet. (NU looks at PU's work)<br>I got it wrong on writing, it should be negative 2. |
| NI | I get negative 2 and another one is 1 |
| | Finally they found out why the results they got were different. |
| NU | I get different values |
| NI | Yes (while aggravating) |

The subject wrote the left derivative and the right derivative as shown in Figure 2.

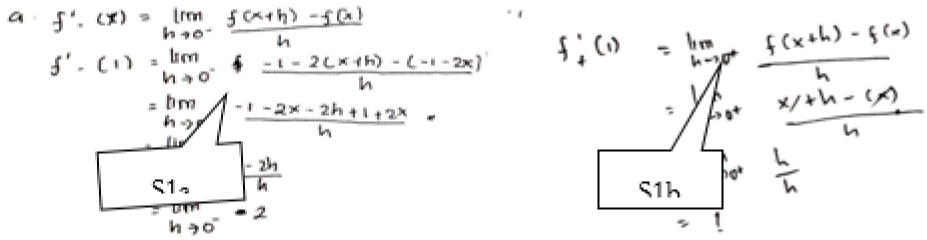



**Figure 2. Values $f'_{-}(1)$ and $f'_{+}(1)$**

In Figure 2, the subject mistakenly took the function to determine the left derivative and did not replace $x$ with 1, namely [S1a], [S2b]. In the process of determining the left derivative and the right derivative, the subject got different values. This is because they were confused in choosing the functions used. Therefore, they solved it by trial and error. In the end, they assigned the first and third functions to determine the left derivative and the right derivative.

Determining the differentiability of the function

According to NU, what should be determined first is the left limit and right limit and their values must be the same. NI agreed with this. Meanwhile, PU said that they must first determine which function to be used. NU and PU chose the function that they would use by guessing. NU said the possibility of the function $f(x) = x$ (pointing to the question sheet), by reason of the interval in the function $x > 1$. The subject then discussed to be able to answer the question in part b. The discussion transcript is given in Table 3.

**Table 3. KITIb Discussion Transcripts**

| Interview | Discussion Transcript |
|---|---|
| NU | Is $f$ differentiable? |
| NI | The left and right limits must be the same |
| NU | But they are not the same thing. |
| PU | What does it mean to be differentiated? |
| The subjects | looking at each other |
| NU | The limit value is present |
| NI | Left and right derivations are present and equal |
| PU | These exist but these are not the same |
| NU | If it is continuous, the limit and function values must be the same. If it is differentiable, what is it? |
| NU | The left and right limits I get are not the same |
| NU | It means it is undifferentiated |
| Nur | at $x = 1$ is undifferentiated because the limit values are not the same. |
| Researcher | What at other point did you test? |
| NU | It will be intervals $-1 \leq x \leq 1$. Is it okay in $x = 0$, ma'am? |
| Researcher | Please try |
| PU | at $x = -1$, ma'am |
| PU | To determine $f$ differentiation, we will use $x = -1$ |
| NI | To test the right side we use $f(x) = x^2$ |
| NU | We use the x squared function, right? |
| NI | Yes |
| NU | From left and right, using the $x$ quadratic function? |
| | They began to discuss |
| NI | Writing while reading what she wrote |
| PU | They said, it should be spelled out $2x + h$. |
| NU | Its derivatives is $2x$ |
| NI | Yes, then substituted $x = -1$ |
| PU | The value is negative 2 |
| NI | $x = -1$Substituted to the function |
| | For the left-2, ma'am |
| Researcher | Right derivation? |
| PU | It is differentiated because it is equal in value |

Based on their discussion in Table 3, the subject of PU said to determine where it is $f$ differentiable, we must use the point $x = -1$. This is because at the point the function is not differentiable$x = 1$.

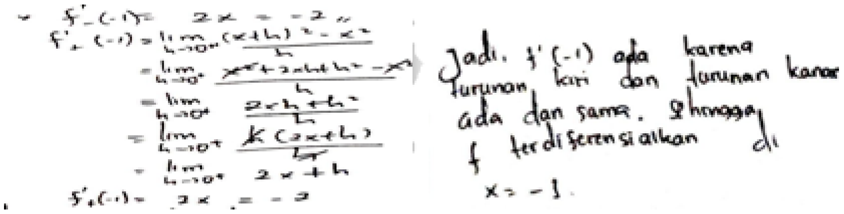



**Figure 3. Determining whether $f$ is differentiable**

NU mentioned that the value of the left derivative was equal to two, as well as the value of the right derivative. The three subjects, PU, NU and NI, could make conclusions from Figure 3. However, the three subjects have not consistently stated the terms left limit or left derivative and right limit or right derivative. It can also be seen that this subject group has not been consistent in choosing the function used.

**Determining whether the function is continuous?**

In the beginning, the three subjects forgot the condition of a function that is said to be continuous. They also found that it is difficult to determine the function and point to be used. Is the point at $x = 1$ or at $x = -1$ be used?

**Table 4. KITIc Discussion Transcripts**

| Interview | Discussion Transcript |
|---|---|
| Researcher | When is a function said to be continuous? |
| NU | Value $f(x)$ equals to limit value |
| NI | Limit value $f(x)$is near to the point |
| Researcher | Try again |
| NU | To define a continuous function, firstly determine the value of $f(x)$ |
| NI | The value $f(x)$ is present and the limit $f(x)$ is close to zero |
| NI | The limit value is there and the left limit and right limit values are the same |
| NU | The addition of the function value equals to the limit value |
| NI | Writing<br>c. Syarat kontinu<br>1. f(x) ada<br>2. lim f(x) ada<br>3. f(x) = lim f(x) |
| NU | c. Syarat kontinu<br>1) f(x) = lim f(x) |
| | They were silent for a long time, looking at the problem. |
| NI | We also pay attention to the intervals |
| Researcher | Ok |
| PU | To see the left and right limits, we use $f(x) = x^2$ |
| | The subject is somewhat confused to determine the function used |
| NI | Just go ahead ma'am, we're taking zeros from the left and zeros from the right |
| Researcher | Let write it down |
| NI | Take on the function $f(x) = x^2$ |
| Researcher | Is it to determine the left limit or the right limit? |
| NU | That means x is close to 0 from the left? |
| | They seem hesitant to decide |

| | |
|---|---|
| NU | I kind of forgot how it is |
| NU | Continuous function must have a limit value, so the function is taken at intervals $-1 \leq x \leq 1$ while pointing to the problem |
| NU | By the way, the left limit from here (points to the function on the question) |
| NU | From the left and right they can see from the interval $-1 \leq x \leq 1$ |
| Researcher | Be careful when determining its function |
| NI | Determining the limit $x$ from the right is used the $x$ quadratic function and $x$ the limit from the left is fixed x squared. |
| Researcher | Take a good look |
| PU | For from the right of the function $x$ and from the left of the $x$ quadratic function |
| NI | I forgot the + and – signs to distinguish left and right at the limit |
| NU | I mistakenly took the function for limit from the left |
| Researcher | Be careful, okay? Pay attention at which point will it be determined? |
| PU | To be determined at point $x = 1$ ma'am |
| Researcher | Okay |
| NU | We have to pay attention to the interval to determine from the left or right, ma'am? |
| NI | Because the left and right limit values are the same, it means that there is a limit of one |
| Researcher | Have all the requirements been met? |
| PU | Not yet ma'am |
| PU | Determining the value of a function |
| NU | The $f(x)$ quadratic function in $x = 1$ |
| PU | The function value equals 1 |
| NU | Continuous in $x = 1$ |
| NU | Oh yes I took it from here (while just pointing the question) |
| PU, NU | We mistakenly determined the point for the left limit and the right limit. |
| NI | Does it take the same function to determine the limit from the left and from the right? |
| Researcher | Try checking again, be careful |
| Researcher | Limit $x$ to one from the right, which function is used? |
| NI | $f(x) = x$ |
| Researcher | How do you write the right and left limit symbols? |
| NI | Writing it down |
| Researcher | Function limit to $x$ go to one on the left, which function is it? |
| NU | From the left, the function is $x$ squared |
| NU | Oh yes I understand, as a basis $x = 1$, it does not $x = -1$ |
| | The interval must be considered |
| PU | Function value equals 1 |
| NI | Because the left limit and the right limit are the same, there is a limit value, which is 1 |
| Researcher | Have the continuous function requirements been met? |
| NU | We'll take that for the function f(x) = x squared |

| | |
|---|---|
| | We use $f(x)$ the same as x squared, we get the value of the function 1 |
| NI | In conclusion, the function is continuous |
| PU | Continuous in $x\ = 1$ |

In Table 4, the subject has made the conclusion that the function is continuous at $x = 1$. Furthermore, the subject was tasked to determine the continuity of the function in Figure 4.

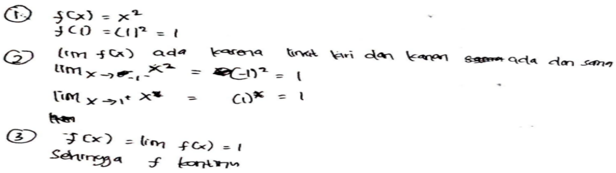
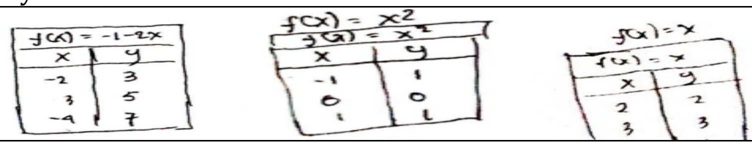
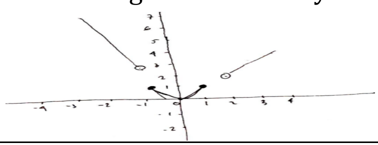
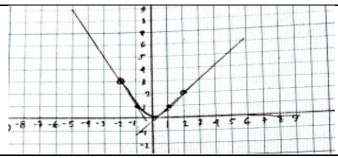


**Figure 4. Determining $f$ continuity**

## Drawing graphs $f(x)$ and $f'(x)$

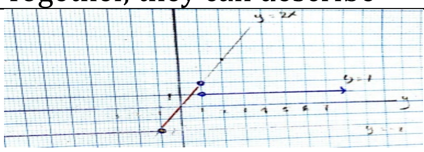
In this section, subject K1 described the three functions separately. NI said that the first function is a straight line, the second is a curve, and the third is also a straight line.

**Table 5. KITId Discussion Transcripts**

| Interview | Discussion Transcript |
|---|---|
| NU | Moving her hands to draw functions |
| PU | Reading the questions |
| NI | The first one described the function$f(x) = -1 - 2x$, in $x < -1$ |
| NI | Take the value of $x$ according to the interval and subsidize it to the function |
| Subject | They inserted those values in the table |
| |  |
| NI | Taking values $x$ starting from -2 because at the first function the interval $x$ is less than negative 1 |
| NU | For $x$ small negative 1, the function will continue to go up |
| NI | $x$ large interval from 1, the chart is straight up, starting from 2 |
| NI | Mentioning the values they take based on the table  |
| PU | Now, we must pay attention to the intervals of each function $f(x) = \begin{cases} -1 - 2x & \text{jika} \quad x < -1 \\ x^2 & \text{jika} \quad -1 \le x \le 1 \\ x & \text{jika} \quad x > 1 \end{cases}$ |
| NI | Specifying a full or empty circle |
| Subject | The second function will meet with the first and third functions |
| |  |

The subject determined the pair of dots $x$ and $y$ and wrote on the Table 5. Furthermore, place a pair of dots on the xy-axis, connect the dots, and form a function image$f(x)$. The subject said that taking the value $x$ starting from -2 because in the first function, the interval $x$ is less than negative 1. The second function is at the point $x = -1; 0\ and\ 1$. The next process they describe$f'(x)$, the subject first determined the derivative for each function$f(x)$. The following is an excerpt of the K1 discussion as given in Table 6.

**Table 6. Discussion Transcripts**

| **Interview** | **Discussion Transcript** |
|---|---|
| PU | We continue illustrating the derivative of the $f$ accent function |
| NU | We determine the derivative, right? |
| NI | Yes, the derivatives |
| N | Derive it directly from its function $-1 - 2x$ |
| Researcher | You should write it completely |
| NI | Writing $f(x) = -1 - 2x$ |
| NU | Like the usual derivative, right? |
| PU | Yes |
| NI | Grafik $f'$ Yesmi<br>$f(x) = -1 - 2x$ $f(x) = x^2$ $f(x) = x$<br>$f'(x) = -2$ $f'(x) = 2x$ $f'(x) = 1$ |
| Researcher | What is the interval like? |
| Subject | The interval is the same ma'am |
| | They are hesitant for $f(x) = 2$ draw, straightly or horizontally. |
| NI | Let's take point (-2, -2) |
| PU | Why did you take -2? |
| NI | We took it from the interval, so (-2, -2), (-3, -2) and so on. |
| Subject | Confused determining the line $f(x) = -2$ |
| Researcher | Give a direction, e.g. $y = f(x) = 3$, the line is horizontal or vertical? |
| PU, NU | Horizontal, at the same time demonstrating with the direction of his hand |
| NI | NI immediately described $y = 3$ |
| Researcher | Then where is $= -2$ ? |
| Subject | Showing where is $y = -2$ on the number line |
| Subject | We weren't careful enough, ma'am (All laughing) |
| | Subject began to describe f accent |
| NU | Putting the coordinate points that they have discussed. |
| Researcher | Draw it in one okay? |
| NU | For $f$ accents, there are three graphs, ma'am |
| PU | Yes. |
| Subject | Together, they can describe |
| |   |

**Task 2**

The second task has similar concepts with the first task. The difference lies in a few questions. This conducts to see how the subject constructs the derived object in their discussion.

Determining the left derivative and the right derivative

The same thing with task one, the subjects were still confused in interpreting $f'_{-}(4)$ and$f'_{+}(4)$, the number 4 in the function is interpreted by them with the fourth derivative. They felt doubt in determining which function should be used and the interval to be used to determine the derivative of the function. PU said for the left derivative, they utilized the second function. NU said the third function was in the form of a fraction. The subjects tried the third function as shown in Figure 5.

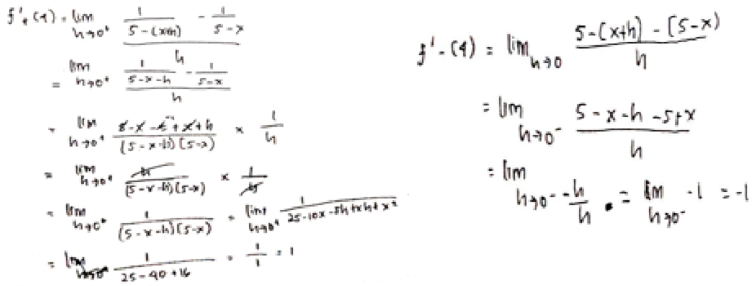
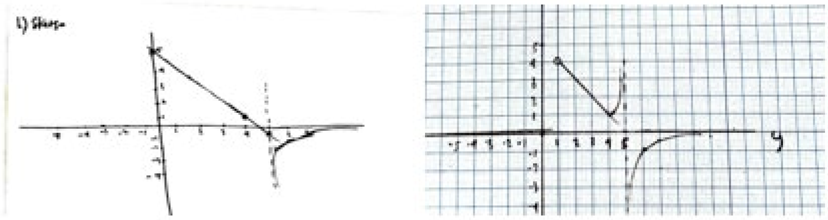


**Figure 5. Values of $f'_{-}(4)$ and $f'_{+}(4)$**

**Drawing $f(x)$**

When the subjects drew the graph of the function $f(x)$, the process they did the same as with task 1. This was conducted by taking several points and made pairs of coordinate points by writing on paper. Subject PU took the point $x = 0$ then the value of $y = 5$; if $x = 4$ then $y = 1$; for $x = 5$ then $y$ was undefined. Subject NI said the a dotted line was drawn at $x = 5$. He gave the reason because y was undefined at $x = 5$ as shown in Figure 6 part (a).



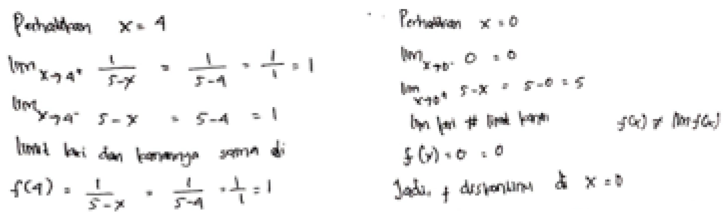
(a) (b)

**Figure 6. Graphics $f(x)$**

Since the picture in part (a) was incomplete furthermore, the researcher asked a question. What should you do? They responded by saying to take some more points for the three functions. Moreover, the subjects completed their drawing by taking more points as shown in Figure 6 part (b). Subjects drew the graph of $f(x)$ by determining pairs of points $(x, y)$ and connecting the points. At $x = 5$, it is depicted with a dotted line called the asymptote. When determining the continuity, subject NU said if it is seen from the figure, the function is not continuous. Subject PU agreed and said it was discontinuous at $x = 5$.

Determine where the discontinuous function is

The subjects were confused in determining the discontinuous function. NI said the continuous function was at $x = 4$, then wrote the subject to be in $x = 4$ and $x = 0$



(a) (b)

**Figure 7. Discontinuous function**

In Figure 7 part (a), the subject obtained the value of $\lim_{x \to 4^+} \frac{1}{5-x} = \lim_{x \to 4^+} (5 - x) = 1$. Subject NU stated $f(4) = 1$. As a result, they concluded that the function is continuous at $x = 4$. Furthermore, in Figure 7 part (b) the subject wrote $\lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x)$. According to the subject NI $f$ is discontinuous at 0, likewise for $x = 5$. Since $f(5)$ is not defined, $f$ is discontinuous at 5.

Based on the subject's work in Figure 8, they mentioned that at the point $x = 0$, the function discontinuous. Furthermore, they said the function was undifferentiated at $x = 4$ because the values of the left derivative and the right derivative are not the same.

d) f tidak terdiferensialkan di $x = 4$ karena turunan kiri dan turunan kanan $f'(x)$ tidak sama

**Figure 8. The subject's conclusion about the undifferentiated function**

In both tasks, it can be seen that the subject has mentioned keywords in each task and visual mediators, both of which are used to construct mathematical objects.

**Task 3**

NU, PU and NI read the given questions one by one and they describe the information on the questions.

**Table 7. KIT3 Discussion Transcripts**

| Interview | Discussion Transcript |
|---|---|
| Researcher | Please review one by one, starting from part a! |
| | How do I remove the symbol for the absolute value? |
| PU | There are two possibilities (pauses). |
| | $x$ for $x$ large or equal to zero and $-x$ for $x$ small and zero |
| Researcher | Let to write it down |
| PU | Writing<br>a) $f(x) = \lvert x \rvert \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$ |
| Researcher | How to determine a continuous function? |
| NI | The absolute value function is said to be continuous |
| PU | Left limit and right limit values are the same |
| NU | The value of the function is also the same |
| NI | Absolute value function has no derivatives? |
| PU | We use the definition of derivative |
| | For the left derivative, change $\lvert x + h \rvert$ to - (x+ h) and |
| NU | For the left derivative, change $\lvert x + h \rvert$ to (x+ h) |
| NI | Writing<br>$f'_-(0) = \lim_{h \to 0^-} \frac{-(x+h) + x}{h} = \lim_{h \to 0^-} \frac{-x-h+x}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1$<br>$f'_+(0) = \lim_{h \to 0^+} \frac{x+h-x}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1$<br>$f'_-(0) \neq f'_+(0) \rightarrow$ tidak mempunyai turunan |
| NI | The problem in part b has never been taught. |
| PU | We see the symbol just now, ma'am |
| NI | We'll continue to part c, ma'am |
| NU | Writing the definition of the function part c |
| | b. $f(x) = \lvert x-1 \rvert \begin{cases} -x+1, & \text{jika } x < 1 \\ x-1, & \text{jika } x \geq 1 \end{cases}$ |
| NU | Non-continuous function |

| | |
|---|---|
| PU | The function is continuous |
| Researcher | Take another look |
| NI | The limit from the right and the limit from the left are the same value |
| NU | Oh yes continuous |
| PU | It does not have derivatives, because the left and right derivatives are not the same |
| Researcher | How about Part D? |
| NU | The left and right limit values are the same, namely zero and the value of the function equals zero |
| PU | The function is continuous |
| NU | Has derivative at x =0 |
| Researcher | Part E |
| NI | We need to determine left and right limits |
| NI | e. $f(x) = x\|x\| \begin{cases} -x^2 & , x < 0 \\ x^2 & , x \geq 0 \end{cases}$ |
| NU | Left limit = 0 and right limit = 0, function = 0 |
| NI | The function is continuous |
| Researcher | Does it have a derivative? |
| PU | The function has a derivative, the left hand value and the right hand value are the same. |
| NI | The function in part E has a derivative |
| Researcher | Parts F? |
| NI | Function has a derivative |
| PU | From the question, the function is also continuous |
| NI | Functions in section G |
| PU, NU | there is another absolute value (both laughs) |
| NI | Writing the function of part F<br>f. $f(x) = \frac{x}{\|x\|} \begin{cases} -1 & , x < 0 \\ 1 & , x \geq 0 \end{cases}$ |
| NU | The first function$f(x) = -1$, and $f(x) = 1$ |
| NU | This function is not continuous, because the values are different |
| PU | The right limit and the left limit are not the same |
| Researcher | Does it have a derivative? |
| NI | $f'(x) = \lim_{h \to 0} \frac{\frac{x}{\|x\|} + h - \frac{x}{\|x\|}}{h}$<br>$f'_+(x) = \lim_{x \to 0^-} \frac{\frac{x}{x} + h - \frac{x}{x}}{h} = \lim_{x \to 0^+} \frac{h}{h} = 1$<br>And $f'_+(x) = 1$, It has no derivative. |
| Researcher | Based on what you have specified above, what is the condition of a function that has derivatives? |

| | |
|---|---|
| Subject | Pay attention and recheck the questions one by one |
| NU | It's continuous but has no derivatives? |
| subject | Grouping derived functions, functions that do not have derivatives. |
| NU | If it had a derivative, it would be continuous |
| NI | Really? |
| NU | Every function that has a derivative must be continuous |
| NI | Functions that have derivatives, if the left derivative and the right derivative are the same |
| Researcher | Are you sure NI? |
| NI | Yes, ma'am |

An excerpt of the subject's discussion which presented in Table 7 is about the characteristics of functions that have derivatives. Here is given the narratives subject about Part a). The NI said the function has no derivative value. PU stated that the left limit value and the right line value are the same, while NU said the function value must be the same. They were asked to double-check the answer. NI corrected the answer, saying that the absolute value function in part a) has no derivatives. Furthermore, NI and NU believe that function$f(x) = |x - 1|$ is continuous. PU also added that the function had no derivatives, because the left and right derivatives were not the same value. They added that they often encountered questions d and f and they concluded that the function was continuous and had derivatives.

In the function of question b, NU said that this function was continuous because the left limit value was equal to zero and the right limit was equal to zero, and so the function was $f(0) = 0$. PU mentioned that the function had a derivative if the left derivative and the right derivative are the same value, which is zero at the zero point. The function $f(x) = \frac{x}{|x|}$, was not continuous and had no derivative x = 0. This function was not continuous, because the values were different (NU) and this function had no derivatives (NI).

To answer the question on the characteristics of a function that has derivative, the subjects checked one by one and they grouped continuous functions and non-continuous functions. NU asked "Do functions that have derivatives definitely continuous?" NI answered that functions have derivatives if the left derivative and the right derivative are the same. They agreed to say that the characteristics of a function that has a derivative, namely the left derivative and the right derivative, have the same value as shown in Figure 9.

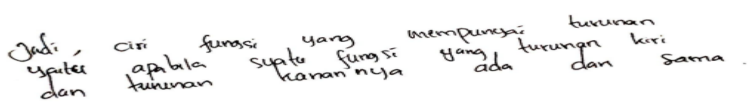



**Figure 9. Function features that have derivatives**

The subjects of the cognitive style field dependent (K1) mentioned the keywords contained in tasks 1 to 3 and the visual mediator that appeared in the form of numbers, algebraic symbols, and graphs used to construct mathematical objects. There were also physical movements when they were thinking. The visual mediators created when presenting data are the basis for analyzing and interpreting (Rahmatina et al., 2022).

**Description of Constructing Mathematics Objects in Derivation of the Cognitive Style Field Independent (K2)**

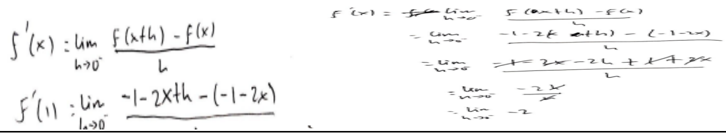
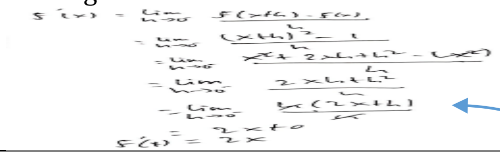
The subjects in the field independent group consisted of three students. They are WA, AM, and KIF. The tasks given to them were carried out by discussion.

**Task 1**

**Determining $f'_{-}(1)$ and $f'_{+}(1)$.**

Before working on task 1a), all subjects were asked to read the problem aloud. There were some symbols mispronounced by them, for example $f'_{-}(x)$ and $f'_{+}(x)$. Moreover, they started working on the problem in part a). The subjects were confused to determine the function used.

**Table 8. K2T1a Discussion Transcripts**

| Interview | Discussion Transcript |
|---|---|
| Researcher | Have the terms left derivative and right derivative been familiar to you? |
| Subject | Yes, ma'am (while nodding). |
| Researcher | Please discuss question 1.a), what do you need to do first? |
| WA | If the $f$ accent is negative, it means that the limit from the left is meaningful, we must first determine $f(x)$. |
| | First, the function is taken $f(x) = -1 - 2x$ if $x$ is less than negative 1 |
| Researcher | How about AM and KIF? |
| AM | Yes ma'am, the same |
| Subject | Start writing down what they mean |
| WA | Each $x$ is changed to $x + h$ |
| | The subjects can determine $f'_{-}(1)$ and $f'_{+}(1)$ |
| WA |  |
| Researcher | Be careful with the function that will be used (reading questions and emphasizing at intervals) |
| WA | It means we are wrong in taking the function |
| | This one, ma'am (while pointing to the second function) |
| KIF | For the right derivative, we take the third function. |
| AM | Writing  Since lim h tends to 0 then the value becomes $2x + 0$ |
| WA | How about the symbols we use? |
| | They are still hesitant to use the symbols of the left derivative and the right derivative. |
| AM | Left derivative equals 2 |
| AM | We just understood that he was wrong in taking on the function |
| AM | If the first function is taken, it means negative one from the left, right, ma'am? |
| Subject | Function used for 1 is from the right? |
| WA | Third function |
| | Subject starts working $f'_{+}(1)$ |
| WA | By taking the 3rd function, the derivative value is one ma'am |

Based on table 8, the subject mistakenly obtained the value of the function $f'_{-}(1)$ and $f'_{+}(1)$. This is due to the inaccurate determination of the function. Initially, to determine the value $f'_{-}(1)$, they utilized the function $f(x) = -1 - 2x$. WA said there was an error in function retrieval. They used the first function. Furthermore, he stated that to determine $f'_{-}(1)$, the second function should be used. They discussed again and wrote it down as in the Figure 10 and repeated their work.

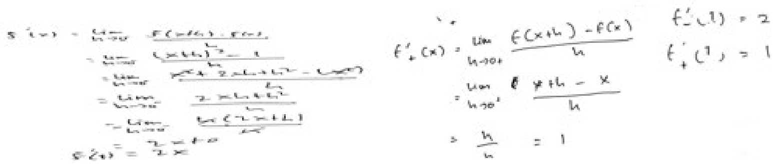


**Figure 10. Determine the value $f'_{-}(1)$ and $f'_{+}(1)$**

The subject seemed confused in taking functions, intervals, and using symbols of the first derivative and the second derivative.

**Determining at what point it $f$ is differentiated**

**Table 9. K2T1b Discussion Excerpt**

| Interview | Discussion Transcript |
|---|---|
| Researcher | Where is f differentiable? |
| WA | If the differentiated function has a derivative.<br>Like $f'(x) = 0$. |
| Researcher | What do you mean by $f'(x) = 0$. |
| WA | It means that the left and the right derivatives are the same or $f'(x)$ exist |
| Researcher | Alright. No $f'(x) = 0$<br>Where f is differentiated? Is it at x =1? |
| WA | No, ma'am.<br>Because at x =1, the value of the left derivative =2 and the right derivative =1, so it is not the same value |
| Researcher | So at what point is the function differentiable? |
| AM | at $x = -1$ |
| Researcher | Show it! |
| WA | Derivative value from the left equals negative 2 |
| AM | What is this function again, ma'am? |
| Researcher | From what do we take the function from left and right? |
| WA | From its interval, ma'am |
| AM | It means $f$ is differentiable in $x = -1$ |

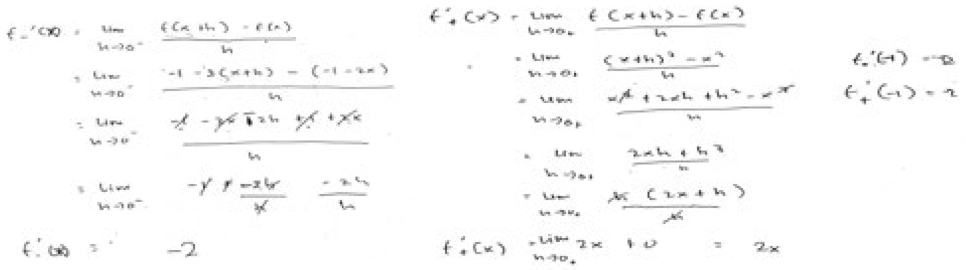
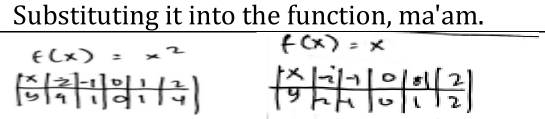


**Figure 11. Specifying whether is $f$ differentiable**

In Table 9, the subject said that $f$ was differentiable in $x = -1$ with a negative value of 2. However, based on Figure 11, they got a value of -2, but the function taking was not yet appropriate.

Is $f$ continuous?

**Table 10. K2T1c Discussion Excerpt**

| Interview | Discussion Transcript |
|---|---|
| Researcher | Is $f$ continuous? |
| AM | If the chart is connected/ unbroken |
| WA | In one cartesian diagram, there are three functions |
| Researcher | At the first function how is the shape of the chart? |
| WA | Linear function ma'am |
| KIF | The second graph is a parabola (while showing with hand) |
| AM | The third is linear function. |
| Researcher | Please describe in one cartesian coordinate! |
| Subject | Start to describe |
| Researcher | How would you describe the function? |
| WA | We picked up a few points. |
| Researcher | What do you mean? |
| AM | Substituting it into the function, ma'am. |
| subject |  |

| | |
|---|---|
| WA | These are the dots that help make the graph. |
| AM | Placing all points on the coordinate axis |
| Researcher | AM, connect the dots for one function! |
| AM | I am calculating the value incorrectly. |
| WA | Describing the first function is wrong because of miscalculation of the value. |
| AM | Connecting the points into a broken line |
| Researcher | Why did you draw a broken line? |
| AM | Recreating the image |
| KIF | Still determining the required points |
| WA | Showing the picture |
| | Wait, ma'am, I'll check the scale again. I have made a mistake. |

In Table 10 the subject discussed about a function said to be continuous. According to AM, to determine the continuous function, it can be seen from the graph, whether the graph is disconnected or connected. WA said it could be depicted in one on a Cartesian diagram. The subjects in the field independent group illustrated the graph separately by making dots in the following Figure 12

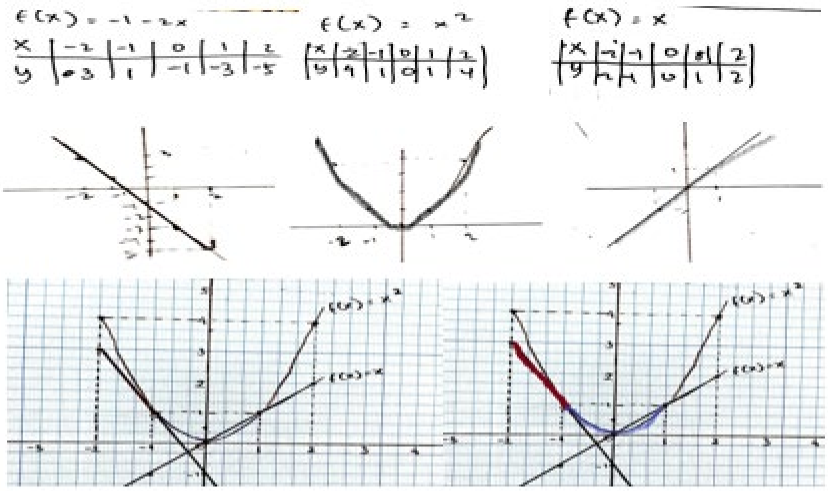



**Figure 12. Determining $f$ continuity**

Furthermore, the subject stated that the interval should be noticed. After that, the subjects fixed the chart by paying attention to the intervals and giving different colors that correspond to the interval limits on the problem. The subjects were asked about the condition of a continuous function. WAL said that firstly, the limit value was there, meaning that the left limit value and the right limit value were the same. However, the subject forgot that the terms of the function value must be equal to the limit value.

**Describing $f'(x)$**

**Table 11. K2T1d discussion excerpt**

| Interview | Discussion Transcript |
|---|---|
| WA | Describing $f$ accents $x$ |
| | We first derive the three functions |
| AM | The derivative of each of these functions right? |
| Researcher | OK |
| KIF | The derivative of the first function is -2 in interval $x$ less than 1 |
| WA | The 2nd function, $f(x) = 2x$, the boundary is fixed |
| Researcher | Please write down the function you are referring to. |
| WA | Writing<br>$f(x) = -1 - 2x \to f'(x) = -2$<br>$f(x) = x^2 \to f'(x) = 2x$<br>$f(x) = x \to f'(x) = 1$ |

| | |
|---|---|
| AM | Writing down the derivative of the function mentioned by the other two $f'(x) = \begin{cases} -2 & \text{jika } x < -1 \\ 2x & \text{jika } -1 \le x \le 1 \\ 1 & \text{jika } x > 1 \end{cases}$ subjects |

In Table 11, the Subjects determined the derivatives for the three functions which given in the problem. The functions that they will draw as follow.

$$f'(x) = \begin{cases} -2\,, & if\ x < -1 \\ 2x, & if\ -1 \le x \le 1 \\ 1, & if\ x > 1 \end{cases}$$

Furthermore, the subjects began to describe the three derivatives of the function. They were confused to describe the function $f(x) = -2$ is vertical or horizontal. After that, they repeated the steps as when they described the continuous function by creating a table and entering the dots.

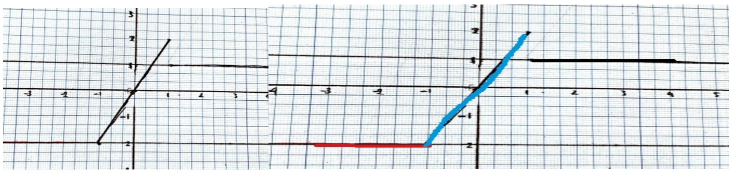


**Figure 13. Graph $f'(x)$**

The subject has described $f'(x)$ in Figure 13 by marking in blue, red and black

**Task 2**

The second task has similar concepts with the first task. The difference lies in a few questions. This was to see how the subject constructed mathematics objects in the material on derivation in their discussion.

**Determining $f'_{-}(4)$ and $f'_{+}(4)$.**

In the second task, AM said that they would determine the left limit and the right limit value. The excerpt of the discussion is demonstrated in Table 12.

**Table 12. K2T2a discussion transcript**

| **Interview** | **Discussion Transcript** |
|---|---|
| AM | We have to determine the left limit and the right limit again, right? |
| WA | Yes, as in question number 1 |
| WA | I'm still confused on the function used to determine the left limit and the right limit |
| AM | This has a function<br>5 less than $x$ is on the left. |
| Researcher | From the left is based on which point? |
| AM | Point 4 |
| | The second function is for the right hand limit |
| Researcher | Also pay attention to chart intervals. |
| AM | Yes, negative one, for the left limit. |

The subject had obtained a left derivative and a right derivative as given in Figure 14.

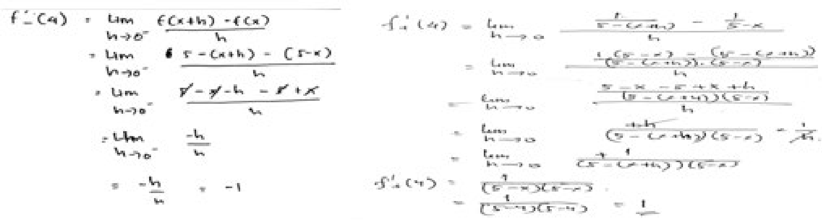


**Figure 14. Determine the value of $f_{-}'(4)$ and $f_{+}'(4)$**

The subjects had answered correctly, but there were some symbols errors. For example, they wrote$f'$_(4), but in the process they still used $x$. The subjects still used the terms left limit and right limit.

**Chart sketch of $f(x)$**

The subjects specified several points for each function. In the third function, when determining the value $f(5)$ they were confused because they got $\frac{0}{0}$. They were silent for some time, then AM said it was undefined, so a dotted line had to be made in $x = 5$, show as Figure 15.

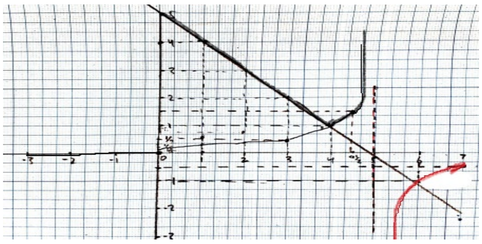


**Figure. 15 Chart sketch of $f(x)$**

Where does $f$ discontinues? AM said that they already checked on $x = 5$, and discontinuous means the chart has broken lines. The other subject said the discontinuous function at $x = 4$.

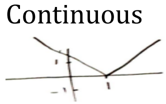
**Where $f$ is undifferentiated**

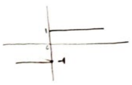
WA said the function has a derivative, meaning that the values of the left derivative and the right derivative were equal. The function $f$ was differentiated at $x = 4$ and $f$ was differentiated at$x = 5$. The subject was still confused to distinguish how to take the differential point on the interval. KIF said the function had no derivative at $x = 0$ so $f$ was undifferentiated at$x = 0$. In both tasks, it can be seen from the discussion that the subject has mentioned keywords in each task and visual mediators, both of which are used to construct mathematical objects.

**Task 3**

In the third task, the field independent group detailed the given functions one by one. According to WA, the function$f(x) = |x|$ is continuous and has no derivative at $x = 0$. WAL said the function has two values, namely $x$ if x is more than zero and negative $x$ if x is less than zero. AM corrected by mentioning $x$ if it is more than or equal to zero. The following are the results of the masculine group discussion, as demonstrated in Table 13.

**Table 13. Function With Information of Formula and its Properties in Task 3**

| | | |
|---|---|---|
| a. $f(x) = \|x\| \begin{cases} x & \text{jika } x \geq 0 \\ -x & \text{jika } x \leq 0 \end{cases}$ | Continuous  | Has no derivatives at $x = 0$ |
| b. $f(x) = \|\|x\|\|$ | discontinuous | Has no derivatives |
| $f(x) = \|x - 1\|$ <br> c. $f(x) = \|x-1\| \begin{cases} x-1 & \text{jika } x \geq 1 \\ -(x-1) & \text{jika } x < 1 \end{cases}$ | Continuous  | Has no derivatives at $x = 0$ |
| $d.\ f(x) = x^2$ | Continuous  | Has derivatives at $x = 0$ |
| $e\ \ f(x) = x\|x\|$ <br> e. $f(x) = x.\|x\| \begin{cases} x.x, & \text{jika } x \geq 0 \\ -x.-x, & \text{jika } x < 0 \end{cases}$ | Continuous  | Has derivatives at $x = 0$ |
| $f.\ \ f(x) = x^2 - 1$ | Continuous  | Has derivatives at x = 0 |

| | | |
|---|---|---|
| 9. $f(x) = \frac{x}{\lvert x \rvert} \cdot \begin{cases} \frac{x}{x} = 1 \\ \frac{x}{-x} = -1 \end{cases}$ | discontinuous  | has no derivative. at x = 0 |
| Ciri² fungsi yg memiliki turunan $f'(x) = 0$ , ada | | |

The three field independent group subjects said that there were two functions in task 3, namely continuous function and non-continuous function in Table 13. The subject of the field dependent Group (K1) mentioned keywords and visual mediators in the form of numbers, algebraic symbols, and graphs used to construct mathematical objects. There was also physical movements when they were thinking. Furthermore, the subject of group K2 made a conclusion demonstrated in Figure 16.

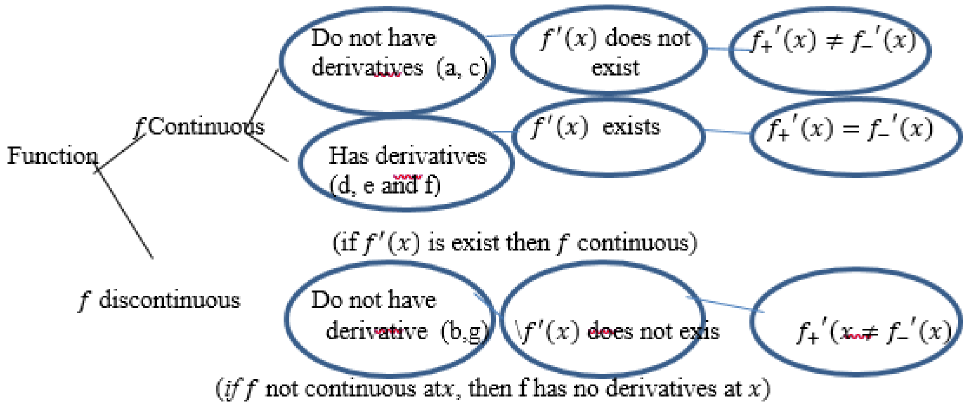



**Figure 16. Conclusion of Task 3 information by K2 Group**

In Figure 16, it can be seen that when the subject defined continuous function information and has no derivatives, there are three different representations. This is also the same for the discontinuous function and has no derivative. Meanwhile, in the non-continuous function information, we have information that its pair does not have derivative.

Based on the discussion data of group-1 (K1) and group-2 (K2) when constructing mathematical objects in tasks 1, 2 and 3 is convergence meaning that the subject can determine the value of the left derivative and the right derivative, but the subject still misread the symbol. Furthermore, the subjects were also doubtful about the function to be used. The subjects could show a differential function, but they were confused when determining the function and determining the interval to be used. Moreover, they could also show continuity of a function, but their narrative often exchanged about the definition of a function which has a limit and a derivative. The subjects could describe the function but they were confused with the role of intervals in each function. To help the subject on the problem in order determine the function used, the strategy that is carried out by starting from the introduction of intervals and functions and symbols that will be used. The investigation reported in this article was a set of findings. This finding shows that there are still many errors in mentioning symbols and doubts in determining the function to be used in completing some tasks. The Figure 16 shows that not all students perform perfectly.

Based on the three tasks that have been discussed by the field dependent and field independent cognitive style groups, it has been seen that there is a mathematical object in the form of signifier along with realization tree. In the case of this derivative, for example in describing two functions, for example $f(x) = -1 - 2x$ and $f(x) = x$ (algebraic expression), a straight line will be formed by using a table by making a pair $(x, y)$. The same applies for the function$f(x) = x^2$. Furthermore, in building mathematical objects in derivative material, it is necessary to have keywords and visual mediato

**Words/keywords.**

The subjects utilized some important words in mathematics, especially those related to derivative material. For example, left derivative, right derivative, left limit, right limit, differensiable, continuity. These keywords can explain the role of language in the understanding of derivatives. The words that students pronounce come from everyday words, for example limit, derivative, approaching which also has a special meaning in mathematical terms. In these derivative tasks, the subject has not been consistent in mentioning the terms "left derivative" and "right limit", they often add the word "next to" or completely mention "right derivative". The same thing when they mentioned limits. Keywords which classified as use words are everyday words are combined with mathematical and non-mathematical language (Mpofu & Pournara, 2018; Mpofu & Mudaly, 2020). Furthermore, all words used in mathematics are called literate word use (Mpofu & Pournara, 2018).

**Visual mediator**

Visual mediator is a visual object which operated as part of the communication process [1]. This visual object is still limited to symbols, about right derivatives and left derivatives, limits, and so on. In this study, the visual mediators that appear are symbols, graphs, algebraic notation, and derivative formulas. Illustrating graphs of $f'(x)$ and $f(x)$ is continuous functions which include as iconic type of visual mediators (Mpofu & Mudaly, 2020). In the meantime, the derivation symbols, differentiation, limits include as symbolic visual mediators (Mpofu & Mudaly, 2020).

**Endorsed Narratives**

Endorsed narratives are often labeled as truth. Sfard dan Lavie (2005) stated that an endorsed narrative is a set of propositions that are accepted and labeled as true by a particular community. In the case of mathematical discourse, the narrative, which can be supported, must be constructed and proven according to a well-defined set of rules. A mathematical discourse is considered to be based on purely deductive relationships between narratives (Tabach &Nachlieli, 2016). According to Sfard (2008), endorsed narratives are known as mathematical theories, which include discursive constructs such as definitions, proofs and theorems. In this analysis, narratives are categorized either as descriptions of entities and the relationships between them, or as narratives about actions with or by entities. In this case, the subjects gave rise to the narrative "functions that have derivation mean differentiable." This is an endorsed narrative in mathematical discourse.

**Routines and realization trees**

Routine is how to look, how to convince, and how to inscribe. It is a pattern of action we recall in a task-situation. Every process in completing derivative tasks requires keywords and visual mediators. The following is an illustration of the completion of differentiable by both groups. In this study, we focused on subjects' way of completing written tasks, speech, and body movements, which could then construct mathematical objects as presented in the Figure 17. The realization routine determines that $f$ is differentiable in $x = -1$.

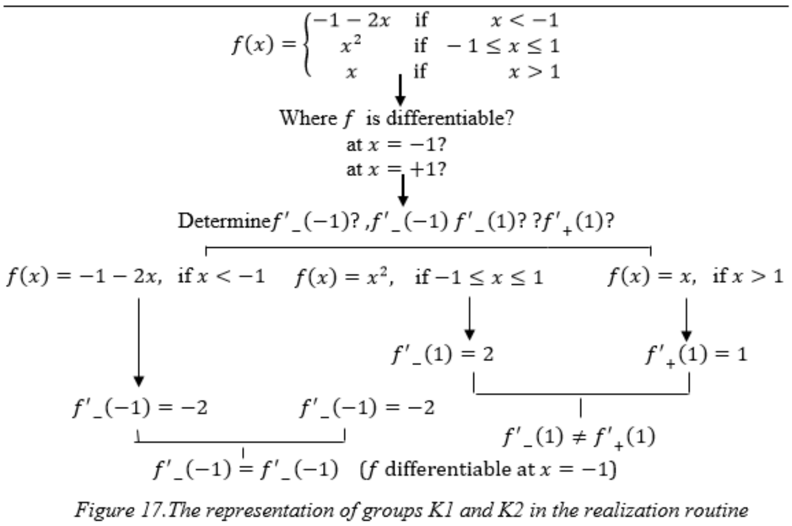



*Figure 17.The representation of groups K1 and K2 in the realization routine*

*Specifying whether $f$ is differentiable at $x = -1$*

**Figure 17.The representation of groups K1 and K2 in the realization routine**

**Specifying whether $f$ is differentiable at $x = -1$**

Furthermore, in Figure 18, the subjects could construct a continuous function based on the definition of the function continuity requirement.

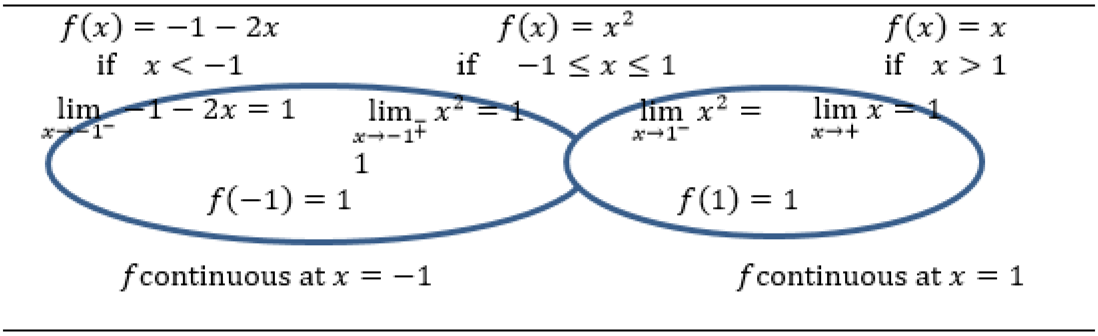
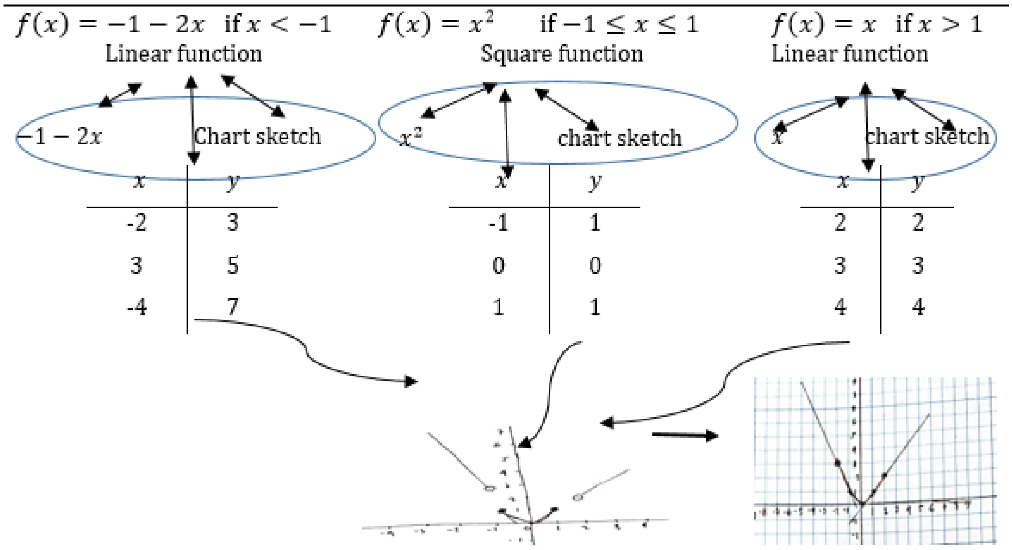



**Figure 18. Continuous function construction according to K1**

In contrast to group 1 which includes a continuous function using the definition of continuity, the K2 group constructed a continuous function as shown in Figure 19.




a) $f(x)$) chart sketch b) $f(x)$ chart sketch

**Figure 19. Chart construction $f(x)$ by K1**

In Figure 19, it can be seen that signifier is a function. For example, the students write $f(x) = -1 - 2x$ is as a marker of abstract objects (numbers or functions). When students can display many different realizations and writing, this is called an abstract object (Sfard 2008, 2018; Mpofu & Mudaly, 2020).

Some of the findings which show similar responses from students in this study is to determine the left derivative, the right derivative, continuity, differentiation, and illustrating the graph. For example, students mixed up between determining functions that have derivatives and continuous functions. When determining the left derivative and the right derivative, they were confused about determining the function to be used, so they made a trial-and-error effort. In this case, the function that should be used to determine the left derivative was used for the right derivative. Determining function $f'_{-}(1)$ and $f'_{+}(1)$ is a mathematical object used to build routines to determine the differentiation of functions in Figure 20 below.

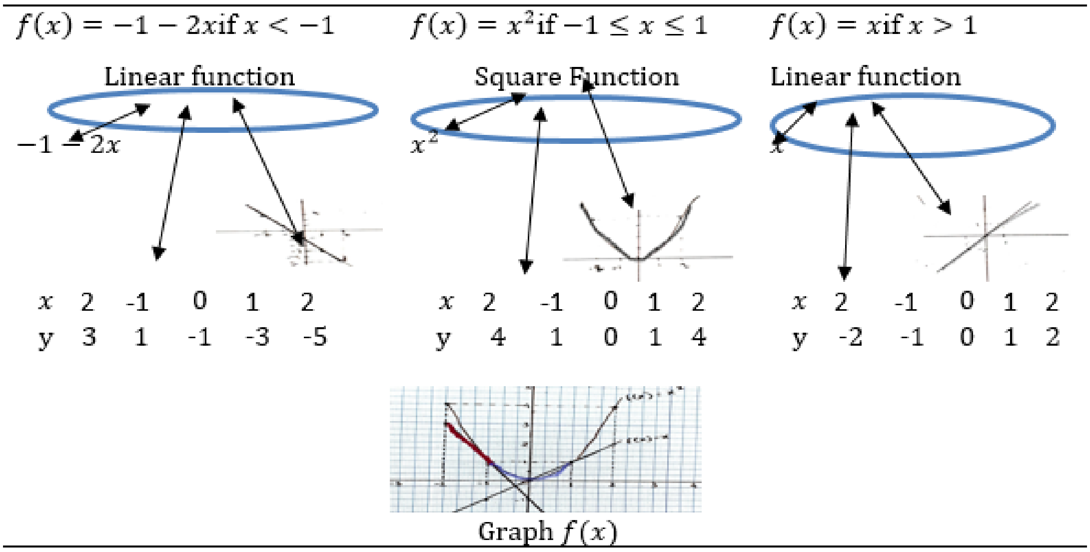



**Figure 20. A realization three of the signifier "continuous function solution according to K2"**

There were also differences in students' perceptions about the terms of left derivative and right derivative. The subject equated the terms left limit and left derivative and right limit and right derivative. This problem was also found in previous research. According to Stewart (2010), a function that is differentiable at $a$ if $f'(a)$ exists. In the KITIb snippet, PU and NU still mentioned the left limit and the right limit. It is the same as in the K2TIb excerpt. In determining the continuous function, the K1 group used an endorsed narrative in the form of a definition while the K2 group used a visual mediator in the form of illustrating the graph. The reason of K2 is the continuity of the function can just be seen from the graph.

## CONCLUSIONS

The current research arose when the first author gave an assignment about derivatives and found that students still had difficulty with symbols and mistaken about terms in mathematics, especially derivatives. Therefore, in this article, students' mathematical objects based on the perspective of commognition were explored (Sfard, 2008). The analysis presented in this article only focused on the discourse of students who completed the assignments in discussion. Mathematical objects that were regularly generated by students were keywords and visual mediators. The tasks given were interconnected, namely determining the left and right derivatives, differentiation, functional continuity, and describing the graph. In completing the task, they discussed with each other.

The field dependent group was found to be inconsistent on the naming of symbols. For example, the naming for the left limit was the same with the left derivative, and vice versa. They were also confused in determining which functions are suitable for the left derivative and the right derivative at the specified point. This certainly makes the routine process a little hampered. Furthermore, the field dependent cognitive style group students in completing tasks 1, 2 and 3 used left-derived, right-derived, differentiated, continuous, discontinuous, graphs, and functions. All these words included keywords and were very helpful in completing the task in this study. The visual mediators in the form of symbols on the three tasks were more widely used. However, there were also errors in reading symbols, for example, symbols $f'_-(1)$, $f'_+(1)$ and $f'_-(4)$, $f'_+(4)$ the errors occur in numbers 1 and 4 and the sign + (states right) and - (states left). Reading symbols must be clear, because misreading symbols will affect further understanding. The hesitation occurred when subjects determine the function. They were more likely to do trial-and-error which causes errors in the calculation process.

At the time of determining the continuous function, the field dependent group also struggled at what point to test, whether to use $x = 1$ or $x = -1$. In Figure 3, it can be seen that this group tested the continuity at the point $x = 1$. Even though at $x = -1$ the function was also continuous. When drawing a function $f(x)$ and $f'(x)$, the purpose is to see the connectedness of the three given functions. At first, the subject drew separately so that the graph could not be interpreted.

Furthermore, the field dependent group utilized dots as mediators, in this case the interval in the predetermined function.

Almost the same as the previous group, the field independent group was also still mistaken in pronouncing the left derivative and the right derivative. In contrast to the previous group, the filed independent group used a visual mediator in the form of a graph in determining continuous function. The process was carried out by drawing the functions separately, then combining them according to the given interval. In the third task, these subjects also explained more on the given information. An explanation of the subject brought up the keywords and visual mediators. In consequence, this forms signifiers as mathematical objects.

## Recommendations

This research explores students' mathematical objects on derivative material. This is conducted by collaboratively and supports the theory of Mathematics, the autopoietic system. Research on mathematical objects by Soejadi's (2010), in the form of facts, concepts, operations and principles have been widely applied. Whereas mathematical objects based on the perspective of commognition are still lacking. According to this perspective, mathematical objects are constructed through discourse, such as numbers, functions, sets, and geometric shapes. Mathematical objects are abstract discursive objects with clear mathematical signifiers, namely the relationship between signifier and realisation. As a result, qualitative research in this field is needed to be developed.

## Limitations

The limitation of this study is the difficulty in exploring mathematical objects in the derivative material task. This is based on the subject's cognition and commognition during the discussion which differed in terms of understanding, pronouncing and using symbols during learning. As a result, it becomes a challenge in future research to be able to apply aspects of cognition, for example the use of keywords "limit", "continuous", discontinuous and others. Students' communication skills need to be maximized as an illustration of their thought process.

This kind of research is still lacking. Meanwhile, Indonesia is directed to use an independent flow that emphasizes communication and discussion in problem solving, especially in mathematics. This is also what must exist in the 21st century generation, which includes ways of working.

## Ethics Statements

This research has received approval from Tadulako University, Palu and was ratified on Mei 2023. This institution is located in the city of Palu, Central Sulawesi, Indonesia.

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## Conflict of Interest

The authors declare that they have no conflicts of interest

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