



## RESEARCH ARTICLE

## Mathematical Modeling of Machine-Tractor Unit Vibrations

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ARTICLE INFO	ABSTRACT
Received: MAY 25, 2026	<p>The article examines the dynamic processes arising during the operation of machine-tractor units (MTUs) based on wheeled tractors with articulated frames coupled with cultivators. The relevance of studying MTU vibrations is substantiated. Particular attention is paid to units based on wheeled tractors with articulated frames coupled with cultivators. The design features of tractors of this type and their influence on the formation of vertical, pitch, and roll vibrations of the unit are analyzed. Taking into account the operating conditions and the nature of external disturbances, a simplified dynamic model is proposed in which the tractor body is considered as a single oscillatory system with equivalent stiffness and damping coefficients. The resulting mathematical model can be used to analyze MTU motion, assess the quality of soil tillage, and substantiate rational design and operating parameters of the units.</p>
Accepted: JUNE 20, 2026	
<b>Keywords</b>	
Machine-Tractor Unit	
Vertical Vibrations	
Pitch Vibrations	
Roll Vibrations	
Oscillatory System	
Dynamic Model	
Lagrange Method	

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### INTRODUCTION

One of the current trends in the development of agriculture is increasing the efficiency of technological processes while simultaneously reducing energy consumption and improving the operational reliability of agricultural machinery [Partko et al., 2020; Groshev et al., 2016]. One of the factors that significantly affects the quality of technological operations and the operational reliability of machinery is its dynamic state during operation [Partko et al., 2019; Groshev et al., 2017]. Vibrations of MTU components lead to reduced uniformity of soil tillage, increased loads on load-bearing components, greater process energy intensity, and a shorter service life of the machinery [Nguyen and Inaba, 2011; Partko et al., 2020; Zhang et al., 2022]. Therefore, the study of MTU vibrations during soil tillage operations is of considerable scientific and practical interest. The choice between a tracked and a wheeled tractor depends on various factors, including the type, condition, and moisture content of the soil, economic considerations, and the tillage technology used. Both types of tractors have their advantages and limitations. Tracked tractors are used for cultivating waterlogged soils because of their high tractive force. At the same time, wheeled tractors are often used for cultivating chernozem soils because of their versatility, mobility, and cost-effectiveness [Arvidsson et al., 2011]. Particular attention in current research is paid to units comprising wheeled tractors with articulated frames and mounted soil tillage implements [Syromyatnikov et al., 2023; Tarasenko et al., 2023]. The design features of tractors of this type significantly affect the nature and parameters of unit vibrations [Gapich et al., 2017]. In particular, the double joint between the front and rear half-frames allows them to rotate relative to each other about the horizontal and vertical axes. However, during straight-line MTU motion, rotation about the vertical axis is virtually absent. Therefore, in the longitudinal vertical plane, the half-frames oscillate as a single system, whereas their roll vibrations may be considered independently [Sidorov et al., 2021].

The running gear and rear hitch of the tractor also have a significant influence on the dynamic properties of the unit. Despite the presence of a sprung front axle and an elastic element in the top link of the hitch system, the sprung and unsprung masses move almost synchronously when the unit operates at relatively low speeds over a field surface with characteristic irregularities. This makes it possible to treat the tractor body as a single oscillatory system with equivalent stiffness determined by the elastic properties of the tires and suspension elements. Similarly, the influence of the elastic element in the top link of the hitch may be neglected when the unit is in its operating position without substantially reducing the adequacy of the model [Partko et al., 2020; Groshev et al., 2016; Nguyen et al., 2011; Partko et al., 2020].

The main sources of MTU vibration excitation are field surface irregularities and the resistance forces acting on the cultivator working elements. These disturbances are random in nature and represent continuously varying dynamic loads [Gapich et al., 2017; Sidorov et al., 2021; Dontsov et al., 2017]. Therefore, investigation of the unit dynamics requires mathematical models that account for both the design features of the MTU and the specific characteristics of external disturbances [Zhu et al., 2014; Zhou J et al., 2017].

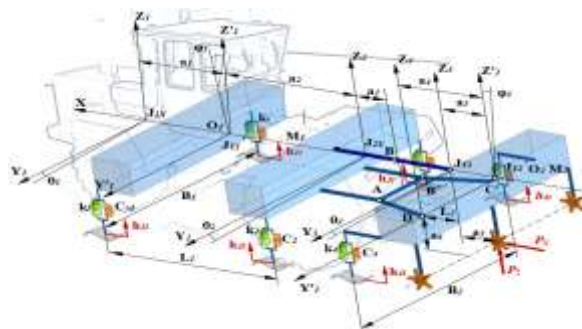
Wheeled tractors are often used to cultivate the chernozem and chestnut soils of Rostov Oblast and Krasnodar Krai because of their known advantages. This study theoretically investigates the vibrations of a machine-tractor unit comprising Ya-250 and BTZ-243K tractors coupled with KNS-3.6, KPM-16, or AKSh-3.6 cultivators during soil milling. The aim of the study is to develop a dynamic MTU model that makes it possible to assess the influence of design and operating parameters on unit vibrations, followed by an analysis of natural frequencies and transfer functions. The following assumptions were adopted in developing the dynamic model: elastic connections are considered linear; damping forces are proportional to deformation rates; the stiffness of the main structural elements is assumed to be infinite; and contact between the wheels and the soil surface is considered continuous during motion. These assumptions are valid under the following conditions: the MTU speed is below 12 km/h; the length of field surface irregularities exceeds 0.5 m; and the frequency of external excitation is below 5 Hz. Outside these conditions, the separate motion of the sprung and unsprung masses must be taken into account.

The load-bearing structure of the Ya-250 and BTZ-243K wheeled tractors is based on an articulated frame supplemented by a sprung front axle. This arrangement provides increased maneuverability and adaptation to terrain irregularities but complicates the dynamic model describing vibrations of the MTU structure.

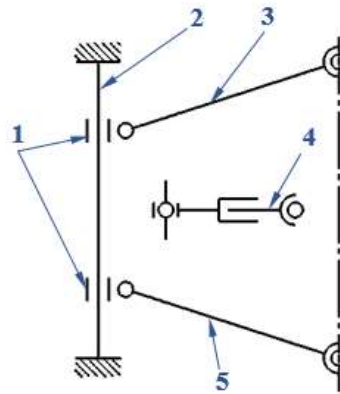
Analysis of the dynamics of similar machine-tractor units shows that a two-mass model is required to describe the vibrations of the sprung axle only under high-frequency external excitation – for example, when moving over short irregularities at high speed.

## METHODS

Under the operating conditions considered – movement of the unit over irregularities of different lengths, both short and long, at low speed – the sprung mass of the tractor body and the unsprung mass of the axle move synchronously. Under these conditions, the tractor body oscillates on springs with equivalent stiffness jointly determined by the front axle springs and the tires of the front wheels. This makes it possible to simplify the calculation model without substantially reducing the accuracy of the system dynamics description (Figure 1).



**Figure 1: Computational Dynamic Model of the MTU**



**Figure 2: Diagram of the Tractor Three-Point Hitch Configuration:**

1 – Lower-Link Head; 2 – Lower Pivot Shaft; 3 and 5 – Right and Left Lower Links, Respectively; 4 – Top Link with a Spring Damper

The rear hitch of the tractor (see Figure 2) is designed as a four-bar linkage mechanism that secures the cultivator at three points. The top link of this device is equipped with a double-acting spring damper, which improves the ability of the cultivator working elements to follow the terrain and reduces impact loads. Nevertheless, when modeling vibrations of the tractor body with a mounted implement in the operating position, the elastic properties of the top link of the hitch system are generally not taken into account.

The main sources of vibrations in the machine-tractor unit (MTU) are surface irregularities and working resistance forces. These external disturbances are continuously acting dynamic loads of variable magnitude and are stochastic and random in nature [Partko et al., 2020; Groshev et al., 2016; Partko et al., 2020; Gapich et al., 2017].

When studying small vibrations of the system relative to the equilibrium position, the following assumptions are adopted: the elastic constraints in the system are considered stationary and holonomic, with a linear relationship between deformation and restoring force; nonelastic resistance in the constraints is proportional to the rates of their deformation; the main assemblies of the unit are modeled as absolutely rigid bodies; and the wheels remain in constant contact with the soil and do not lose contact during vibration [Partko et al., 2020; Groshev et al., 2016].

The following designations are used in the computational model of the MTU shown in Figure 1, which was developed taking into account the above assumptions and the design features of the unit:  $M_1$  and  $M_2$  are the masses of the tractor and cultivator, respectively;  $J_{1x}$ ,  $J_{2x}$ , and  $J_{3x}$  are the moments of inertia of the front part of the tractor, the rear part of the tractor, and the cultivator about the  $x$ -axis, respectively;  $J_{Y1'}$  is the moment of inertia of the tractor about the  $Y1'$ -axis;  $J_{Y2'}$  is the moment of inertia of the cultivator about the  $Y2'$ -axis;  $C_1$  is the radial stiffness of the front tire;  $C_3$  is the spring stiffness;  $C_2$  and  $C_4$  are the radial stiffnesses of the rear tractor tires and cultivator support-wheel tires, respectively;  $L_1$  and  $L_2$  are the wheelbases of the tractor and cultivator, respectively;  $B_1$  and  $B_2$  are the wheel tracks of the tractor and cultivator, respectively;  $a_1$ ,  $a_2$ ,  $a_3$ ,  $a_4$ ,  $a_5$ , and  $a_6$  are the geometric parameters of the MTU;  $h_{1l}$ ,  $h_{1r}$ ,  $h_{2l}$ ,  $h_{2r}$ ,  $h_{4l}$ , and  $h_{4r}$  are the time-varying disturbances from the field acting on the left and right wheels of the MTU;  $k_1$  and  $k_2$  are the damping coefficients of the tire and spring, respectively;  $k_3$  and  $k_4$  are the damping coefficients of the rear tractor tires and cultivator support-wheel tires, respectively;  $P_x(t)$  and  $P_z(t)$  are the working resistance forces.

Let us introduce the equivalent stiffness coefficient of the tractor front support,  $C_{rd}$ :

$$C_{rd} = \frac{C_1 \cdot C_3}{C_1 + C_3}, \quad (1)$$

Let us introduce the equivalent damping coefficient of the tractor front support:



$$\begin{aligned}
2T = & M_1 \left( \frac{a_2 \dot{z}_1 + a_1 \dot{z}_1}{L_1} \right)^2 + J_{y_1'} \left( \frac{\dot{z}_1 - \dot{z}_2}{L_1} \right)^2 + J_{1x} \dot{\theta}_1^2 + J_{2x} \dot{\theta}_2^2 + \\
& + M_2 \left( \frac{a_1 \dot{z}_3 + a_3 \dot{z}_4}{L_2} \right)^2 + J_{y_2'} \left( \frac{\dot{z}_3 - \dot{z}_4}{L_2} \right)^2 + J_{3x} \dot{\theta}_3^2,
\end{aligned} \tag{5}$$

Expression for potential energy:

$$\begin{aligned}
2E = & C_{rd} \left\{ \left[ \left( z_1 + \frac{B_1}{2} \theta_1 \right) - h_{1l} \right]^2 + \left[ \left( z_1 + \frac{B_1}{2} \theta_1 \right) - h_{1r} \right]^2 \right\} + \\
& + C_2 \left\{ \left[ \left( z_2 + \frac{B_1}{2} \theta_2 \right) - h_{2l} \right]^2 + \left[ \left( z_2 + \frac{B_1}{2} \theta_2 \right) - h_{2r} \right]^2 \right\} + \\
& + C_4 \left\{ \left[ \left( z_4 + \frac{B_2}{2} \theta_2 \right) - h_{4l} \right]^2 + \left[ \left( z_4 + \frac{B_2}{2} \theta_3 \right) - h_{4r} \right]^2 \right\},
\end{aligned} \tag{6}$$

Expression for dissipated energy:

$$\begin{aligned}
2R = & k_{rd} \left\{ \left[ \left( \dot{z}_1 + \frac{B_1}{2} \dot{\theta}_1 \right) - \dot{h}_{1l} \right]^2 + \left[ \left( \dot{z}_1 + \frac{B_1}{2} \dot{\theta}_1 \right) - \dot{h}_{1r} \right]^2 \right\} + \\
& + K_2 \left\{ \left[ \left( \dot{z}_2 + \frac{B_1}{2} \dot{\theta}_2 \right) - \dot{h}_{2l} \right]^2 + \left[ \left( \dot{z}_2 + \frac{B_1}{2} \dot{\theta}_2 \right) - \dot{h}_{2r} \right]^2 \right\} + \\
& + K_4 \left\{ \left[ \left( \dot{z}_4 + \frac{B_2}{2} \dot{\theta}_3 \right) - \dot{h}_{4l} \right]^2 + \left[ \left( \dot{z}_4 + \frac{B_1}{2} \dot{\theta}_3 \right) - \dot{h}_{4r} \right]^2 \right\}.
\end{aligned} \tag{7}$$

The generalized forces corresponding to the adopted generalized coordinates are represented as follows:

$$\begin{aligned}
Q_{z_3} &= P_x \frac{a_6}{L_2} + P_z \frac{\alpha_2 - a_5}{L_2}; \\
Q_{z_4} &= -P_x \frac{a_6}{L_2} + P_z \frac{a_5}{h_2}; \\
Q_{\theta_3} &= 0
\end{aligned} \tag{8}$$

After substituting Equations (2), (3), (4), (5), (6), (7), and (8) into Equation (1), the MTU vibrations are described by a system of eight differential equations, which is presented in expanded form in the table 1.

**Table 1: System of Differential Equations for Small Longitudinal and Transverse Vibrations of the MTU**

Generalized Coordinate Number, k	Equations
k=1	$ \begin{aligned} & \frac{M_1 a_2^2 + J_{y_1'}}{L_1^2} \ddot{z}_1 + 2k_{rd} \dot{z}_1 + 2C_{rd} z_1 + \frac{M_1 a_1 a_2 - J_{y_1'}}{L_1^2} \ddot{z}_2 - \lambda Q_1 = \\ & = k_{rd} \dot{h}_{1l} + k_{rd} \dot{h}_{1r} + c_2 \dot{h}_{2l} + k_2 h_{1r}; \end{aligned} $

<b>k=2</b>	$\frac{M_1 a_2^2 + J_{y_1}'}{L_1^2} \ddot{z}_1 + 2k_2 \dot{z}_2 + 2c_2 z_2 + \frac{M_1 a_1 a_2 - J_{y_1}'}{L_1^2} \ddot{z}_1 - \lambda Q_1 =$ $= k_2 \dot{h}_{2l} + k_{np} \dot{h}_{2r} + c_2 h_{2l} + k_2 h_{2r};$
<b>k=3</b>	$\frac{M_2 a_4^2 + J_{y_2}'}{L_2^2} \ddot{z}_3 + \frac{M_2 a_3 a_4 - J_{y_2}'}{L_1^2} \ddot{z}_4 - \lambda q_2 = P_x \frac{a_6}{L_2} + P_z \frac{\alpha_2 - \alpha_3}{L_2};$
<b>k=4</b>	$\frac{M_2 a_3^2 + J_{y_2}'}{L_1^2} \ddot{z}_4 + 2k_4 \dot{z}_4 + 2c_4 z_4 + \frac{M_2 a_3 a_4 - J_{y_2}'}{L_2^2} \ddot{z}_3 - \lambda Q_2 =$ $= k_4 \dot{h}_{4l} + k_4 \dot{h}_{4r} + c_4 h_{4l} + k_4 h_{4r} - P_x \frac{a_6}{\alpha_6} + P_z \frac{a_5}{\alpha_2};$
$q_k = \theta_1$	$J_{1x} \ddot{\theta}_1 + k_{rd} \frac{B_1^2}{2} \dot{\theta}_1 + c_{rd} \frac{B_1^2}{2} \theta_1 = k_{rd} \frac{B_1}{2} \dot{h}_{1l} - k_{rd} \frac{B_1}{2} h_{1r} + c_{rd} \frac{B_1}{2} h_{1l} - c_{np} \frac{B_1}{2} h_{1r};$
$q_k = \theta_2$	$J_{2x} \ddot{\theta}_2 + k_2 \frac{B_1^2}{2} \dot{\theta}_2 + c_2 \frac{B_1^2}{2} \theta_2 = k_2 \frac{B_1}{2} \dot{h}_{2l} - k_2 \frac{B_1}{2} \dot{h}_{2r} + c_2 \frac{B_1}{2} h_{2l} - c_2 \frac{B_1}{2} h_{2r};$
$q_k = \theta_3$	$J_{3x} \ddot{\theta}_3 + k_4 \frac{B_2^2}{2} \dot{\theta}_3 + c_4 \frac{B_2^2}{2} \theta_3 = k_4 \frac{B_2}{2} \dot{h}_{4l} - k_4 \frac{B_1}{2} \dot{h}_{4r} + c_4 \frac{B_1}{2} h_{4l} - c_4 \frac{B_1}{2} h_{4r};$
<b>Constraint Equation</b>	$f = Q_1 z_1 - Q_1 z_2 + Q_2 z_3 - Q_2 z_4 = 0$

**RESULTS**

Using the Laplace transform, we convert the variables into the operator form, assuming zero initial conditions. The system of differential equations presented in the table is written as the following matrix equation:

$$\|A(p)\| \cdot \|z(p)\| = \|B(p)\| \cdot \|h(p)\|, \tag{9}$$

where  $p$  is the Laplace operator;  $\|z(p)\|$ ,  $\|h(p)\|$  the indicated quantities are the column vectors of coordinates and disturbance functions;  $\|A(p)\|$  and  $\|B(p)\|$  the indicated matrices are the matrices of the system operators and disturbance operators.

To solve Equation (9), we multiply it  $\|A(p)\|^{-1}$  from the left by the indicated matrix:

$$\|z(p)\| = \|A(p)\|^{-1} \cdot \|B(p)\| \cdot \|h(p)\|.$$

The transfer-function matrix  $\|\omega'\| = \|A(p)\|^{-1} \cdot \|B(p)\|$  fully determines the dynamic properties of the system. It can be used to determine the MTU vibration parameters under various external disturbances.

**CONCLUSIONS AND DISCUSSION**

In the analysis of MTU vibrations, the dynamic state of the unit is determined by vertical, pitch, and roll vibrations. Their characteristics are significantly influenced by the design features of the tractor and cultivator, rather than solely by field surface irregularities and variable resistance forces acting on the cultivator working elements.

The tractor body can be represented as a single oscillatory system with equivalent stiffness and damping coefficients determined by the elastic properties of the tires, suspension elements, and

sprung front axle. It has been shown that the influence of the elastic element of the top link of the hitch may be neglected when the unit is in the operating position.

A mathematical model of the MTU was developed that accounts for the design parameters of the tractor and cultivator, as well as the random nature of external disturbances. The Lagrange method with redundant coordinates was used to describe the system dynamics, making it possible to derive a system of differential equations for small longitudinal and transverse vibrations of the MTU.

A transfer-function matrix was obtained that fully determines the dynamic properties of the unit under study. The resulting model makes it possible to evaluate MTU vibration parameters under different operating conditions and external disturbances.

The results of the study can be used to develop measures aimed at reducing dynamic loads on MTUs comprising Ya-250 and BTZ-243K tractors coupled with KNS-3.6, KPM-16, and AKSh-3.6 cultivators.

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