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RESEARCH ARTICLE

Reconstruction of the Locally Inhomogeneous Elastic Modulus of a Bar in Two-Dimensional Anti-Plane Vibration Problems

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ABSTRACT

Dynamic problems in solid mechanics related to the analysis of vibrations and wave processes in extended structural elements have traditionally occupied a central place in applied mathematics and mechanical engineering. Among the full range of acoustic fields, anti-plane, or purely shear, vibrations of a bar are of particular importance. In this case, the displacement vector has only one nonzero component directed along the longitudinal axis of the elastic body, while the strain field itself depends only on two transverse coordinates. The mathematical rigor of the anti-plane shear equations, which reduce to elliptic differential equations with variable coefficients, makes this model a fundamental basis for developing and verifying new methods of ultrasonic tomography of media (Babeshko and Ratner, 2023; Vatulyan and Mnukhin, 2021). Under real operating conditions, structural elements are subjected to nonuniform mechanical, thermal, and radiation effects that cause degradation of the material structure. At the early stages of damage accumulation, even before macroscopic cracks or voids form, local zones potentially prone to fracture arise in the material and are characterized by a local decrease in stiffness properties (Vatulyan and Nedin, 2021). Detection of such zones belongs to the class of inverse coefficient problems in mechanics, which are mathematically characterized by substantial nonlinearity and Hadamard ill-posedness, that is, instability of the solution under small perturbations of the input data (Kollas and Tikhonov, 2024). For the numerical implementation of reconstruction procedures, the finite difference method (FDM) is the most transparent and effective tool (Samarskii, 2023). In contrast to projection approaches, finite difference approximations make it possible to establish explicit algebraic relationships between the unknown discrete distribution of the shear modulus at the nodes of the computational grid and the recorded wave field, which significantly simplifies the calculation of gradients of objective functionals. This study is devoted to an inverse coefficient problem, namely mathematical modeling of the reconstruction of the spatial distribution of the elastic modulus of a bar with a local inhomogeneity based on the analysis of the measured characteristics of a two-dimensional anti-plane wave field on the boundary of the object under study.

INTRODUCTION

The relevance of the study is determined by the need to improve the operational reliability of extended structural elements through accurate detection of hidden defects using anti-plane shear vibrations, which are highly sensitive to stiffness changes. The development of a stable finite difference algorithm for solving the inverse coefficient problem makes it possible to effectively identify the geometric and physical parameters of local elastic modulus inhomogeneities under input measurement noise, which is critically important for the development of preventive non-destructive testing.

The first area is associated with the development of classical gradient and iterative methods for minimizing residual functionals. Studies by Achenbach (2023) and Boyd and Guzina (2022) show that, for weak inhomogeneities, the Born approximation is effective, as it converts the problem into a linear integral formulation. However, for local defects with a high gradient of stiffness reduction, researchers have to use modifications of Newton–Kantorovich methods and Levenberg–Marquardt schemes. Chen (2025) emphasizes that the use of finite difference schemes (FDM) on uniform grids makes it possible to efficiently compute sensitivity matrices, or Jacobians, for iterative procedures, substantially reducing computing time compared with commercial finite element packages.

The second major area is focused on overcoming the ill-posedness of inverse problems using regularization methods and numerical-analytical reductions. The stability of recovering coefficients of differential equations in the presence of noisy data on the boundary of a bar is considered in detail by Gao (2023). Researchers actively combine first-order Tikhonov regularization with truncated singular value decomposition (TSVD). Studies by Karniadakis et al. (2021) and Nintcheu (2021) successfully implemented an approach based on the boundary integral equation method (BIEM) and discrete Green's functions, in which the effect of a local inhomogeneity is transferred exclusively to the integral over the region of its actual occurrence. This makes it possible to narrow the defect search area in the bar cross-section and ensures algorithm stability at measurement noise levels of up to 5%.

The third and most rapidly developing area over the past three years, 2023–2026, is the introduction of physics-informed neural networks (PINNs) into defect detection problems (Raissi, 2024). Recent studies by Wang and Takenaka (2022) and Yuan (2024) use deep architectures in which the regularizer is the two-dimensional Helmholtz equation itself, with an unknown variable coefficient, embedded in the loss function. Despite the high approximation speed, Zhang (2026) notes that PINN approaches are often inferior to the classical finite difference method in the accuracy of locating sharp jumps in the elastic modulus at defect boundaries.

Summarizing the current state of the problem, it can be concluded that the construction of hybrid finite difference algorithms combining conservative difference schemes for the mechanics of inhomogeneous media with fast gradient regularization methods remains an open and relevant problem in applied mathematics.

History of the Origin and Development of Problems on Torsional Vibrations

The historical path of research into torsional vibrations and wave processes in cylindrical bodies dates back to the classical works of C. Coulomb and the full-scale experiments of G. Kirchhoff in the eighteenth and nineteenth centuries, which laid the foundations of the mechanics of rod torsion. The mathematical foundation of the dynamic theory of elasticity for extended circular cylinders was formulated in detail in the pioneering works of L. Pochhammer (1876) and C. Chree (1886). They were the first to obtain exact analytical solutions of wave equations for homogeneous isotropic cylinders of infinite length and to construct the classical Pochhammer–Chree dispersion curves, which describe the dependence of wave propagation velocity on frequency.

For a long time, the theory of torsional vibrations developed primarily within the framework of J. Bernoulli's plane-section hypothesis and Saint-Venant's warping theory, which successfully solved applied engineering problems related to the calculation of crankshafts, turbine rotors, and transmission elements. However, the rapid development of technologies in the second half of the twentieth century – the emergence of functionally graded materials and composites, as well as the increasing complexity of the geometry of load-bearing elements – revealed the limitations of classical homogeneous models. This created the need to explicitly account for the spatial inhomogeneity of physical and mechanical properties, including shear modulus and density, and for the variable cross-section of real structures.

At the turn of the century, the focus of scientific research shifted from the analysis of direct dynamic problems to the study of significantly more complex inverse coefficient problems, including acoustic tomography and non-destructive testing. A fundamental contribution to the rigorous mathematical formulation of this field in Russian science was made by the Rostov school of mechanics under the leadership of Academician I. I. Vorovich and Professor A. O. Vatulyan. The methods of operator analysis and integral equations developed by them made it possible to prove fundamental uniqueness theorems for the reconstruction of elastic profiles, transforming qualitative engineering assumptions into a rigorous mathematical discipline.

Under modern conditions, when structures operate under extreme thermal and mechanical regimes, local zones of thermal or fatigue-related stiffness degradation inevitably arise in the material of cylindrical elements even before visible macrocracks appear. Timely detection of such hidden inhomogeneities requires the construction of accurate two-dimensional models. Moving from the historical context to a rigorous mathematical description, we formulate the direct and inverse problems of torsional vibrations of a cylinder with a locally inhomogeneous elastic modulus in the longitudinal section.

MATERIALS AND METHODS

Mathematical Formulation of the Problem in a Two-Dimensional Domain

A longitudinal section of a cylinder is considered, namely a two-dimensional domain Ω in the Cartesian coordinate system (x, y), located in the half-plane y ≥ 0. The domain is bounded by the symmetry axis y = 0, the end sections x = 0 and x = L, and the smooth generatrix of the lateral surface profile y = Y(x):

$$\Omega = \{0 \le x \le L,\ 0 \le y \le Y(x)\}$$

The cylinder material is assumed to be isotropic but spatially inhomogeneous, with density and shear modulus being functions of the coordinates. In the presence of a local stiffness defect, the sought elastic modulus is represented as:

where is the nominal modulus value of the defect-free matrix, and is a function describing the local inhomogeneity. The equation of steady-state vibrations with frequency ω has the form:

(1)

– shear modulus, – displacement.

The domain is bounded by a smooth curve, with the boundary rigidly clamped and a load applied on . The boundary conditions are

(2)

Problem (1)–(2), for a known positive is a boundary value problem for determining the displacement. Even for simple domains, the solution of such a problem can be obtained only numerically. The main fundamental theorem of uniqueness and existence for inverse problems of reconstructing elastic profiles in anti-plane vibrations was proved in the foundational works of the Rostov school of mechanics under A. O. Vatulyan using methods of operator analysis and boundary integral equations. In particular, joint results by the authors on the theory of parameter identification were published in the proceedings of international scientific conferences of Southern Federal University (Vatulyan and Gukasian, 2012). The full mathematical basis for inverse coefficient problems, uniqueness proofs, and operator analysis theorems for elastic bodies is presented in the monograph (Vatulyan et al., 2016).

Finite Difference Method in the Direct Problem of Anti-Plane Vibrations

For the numerical implementation of the direct problem (1)–(2) in the two-dimensional domain Ω, a uniform computational grid is introduced with steps h_x and h_y in the x and y coordinates, respectively. Let the grid nodes be denoted as $\left(x_i, y_j\right)=\left(ih_x, jh_y\right)$, where $i=0,1,\ldots,N_x$ and $j=0,1,\ldots,N_y$. The discrete values of the sought displacement field and the variable shear modulus at the grid nodes are written as $u_{i,j}=u\left(x_i, y_j\right)$ and $\mu_{i,j}=\mu\left(x_i, y_j\right)$. The transition from the continuous differential model to the discrete model is carried out by approximating spatial derivatives with difference operators. When expanding the differential operator with respect to the x coordinate in the form $\mu\frac{\partial^2 u}{\partial x^2}+\frac{\partial \mu}{\partial x}\frac{\partial u}{\partial x}$ a second-order accurate central difference scheme is applied at the main grid nodes:

$$\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)\Bigg|_{i,j}\approx\mu_{i,j}\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h_x^2}+\frac{\mu_{i+1,j}-\mu_{i-1,j}}{2h_x}\frac{u_{i+1,j}-u_{i-1,j}}{2h_x}$$

Similarly for y. Applying these difference operators to (1)–(2), we obtain a discrete linear equation for the internal nodes of the computational grid, representing the finite difference scheme of the direct problem

$$\mu_{i,j}\left(\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h_x^2}+\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{h_y^2}\right)+\frac{\left(\mu_{i+1,j}-\mu_{i-1,j}\right)\left(u_{i+1,j}-u_{i-1,j}\right)}{4h_x^2}+$$
$$+\frac{\left(\mu_{i,j+1}-\mu_{i,j-1}\right)\left(u_{i,j+1}-u_{i,j-1}\right)}{4h_y^2}+\kappa^2u_{i,j}=0 \tag{3}$$

The transition from the continuous boundary value problem to a system of linear algebraic equations is carried out by approximating differential equation (1) at all internal nodes of the computational grid and discretizing boundary conditions (2) on its boundaries. To construct a unified system of equations, global indexing is introduced. The two-dimensional array of unknown displacements $u_{i,j}$ is expanded into a global vector U of dimension $M=\left(N_x+1\right)\times\left(N_y+1\right)$. Node numbering defines the relationship between the two-dimensional index (i,j) and the one-dimensional global row index k:

$$k=i+j\left(N_x+1\right)+1,\ i\in\left[0,N_x\right],\ j\in\left[0,N_y\right]$$

For all internal points of the computational domain, where $i=1,2,\dots,N_x-1$ and $j=1,2,\dots,N_y-1$, standard rows of the system of linear algebraic equations are formed. From the previously obtained five-point finite difference scheme, the canonical equation for the k-th row of matrix A takes the form:

$$A_{k,k-(N_x+1)}u_{k-(N_x+1)}+A_{k,k-1}u_{k-1}+A_{k,k}u_k+A_{k,k+1}u_{k+1}+A_{k,(N_x+1)}u_{k+(N_x+1)}=F_k \tag{4}$$

where the grid coefficients for each internal node (i,j) are determined by the following relations:

$$A_{k,k-1}=\frac{\mu_{i,j}}{h_x^2}-\frac{\mu_{i+1,j}-\mu_{i-1,j}}{4h_x^2}$$

$$A_{k,k+1}=\frac{\mu_{i,j}}{h_x^2}+\frac{\mu_{i+1,j}-\mu_{i-1,j}}{4h_x^2}$$

$$A_{k,k-(N_x+1)}=\frac{\mu_{i,j}}{h_y^2}-\frac{\mu_{i,j+1}-\mu_{i,j-1}}{4h_y^2}$$

$$A_{k,k+(N_x+1)}=\frac{\mu_{i,j}}{h_y^2}+\frac{\mu_{i,j+1}-\mu_{i,j-1}}{4h_y^2}$$

$$A_{k,k}=\kappa^2-2\mu_{i,j}\left(\frac{1}{h_x^2}+\frac{1}{h_y^2}\right)$$

At the boundaries of the computational grid, the standard finite difference equation of the original differential equation becomes inapplicable, since the difference stencils extend beyond the domain Ω. At these nodes, the equations are completely replaced by finite difference analogues of the corresponding boundary conditions, which makes it possible to exclude all nodal points located on the boundary of the domain.

Numerical Implementation of the Direct Problem

For clarity, let us consider a specific physical example of numerical modeling of a steel bar of rectangular cross-section with a local structural defect. The bar length is $l = 1.0$ m, the height is $H = 0.4$ m, the vibration frequency is ω, at which the frequency parameter is $\kappa^2 = \frac{\rho\omega^2}{10^9} = 15.0\,м^{-2}$, and the load amplitude is $P_0(y) = 50МПа = 0.05ГПа$, the law of variation of the shear modulus is

$$\mu(x,y) = \begin{cases} 2.0\,, (x,y) \in \Omega_{def} \\ 2.0 + x^2 + y^2, (x,y) \in \Omega, (x,y) \notin \Omega_{def} \end{cases}$$ GPa,

where $\Omega_{def} = \{(x,y) | 0.4м \le x \le 0.6м,\ 0.1м \le y \le 0.2м\}$

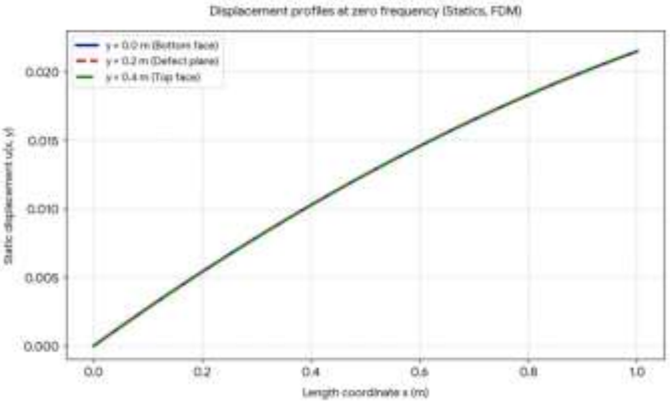


Figure 1: Solution of the Direct Problem by the FEM

Figure 1 presents the solution of the direct problem based on the FEM; the same graph shows the results of determining the displacement $u(x,y)$ in different cross-sections.

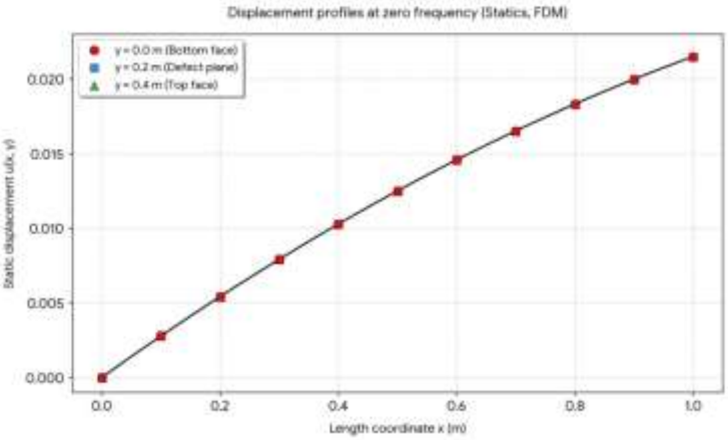


Figure 2: Results of Solving the Direct Problem Using the MDA

(method of difference approximations). Along the axis y the value $u(x,y)$ in different cross-sections is shown.

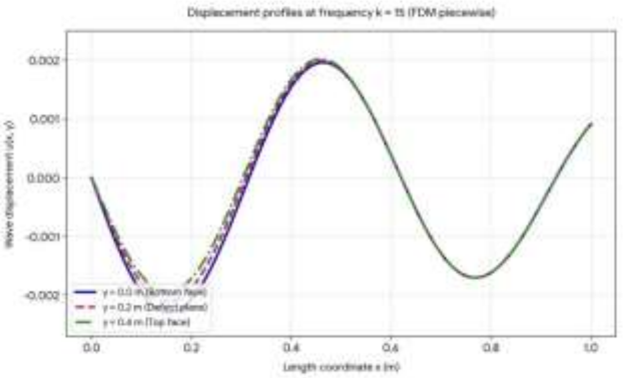


Figure 3: Distribution of Wave Displacements $u(x,y)$ in Cross-Sections of an Elastic Bar at the wave parameter $\kappa = 15$ m⁻¹, obtained by the FEM

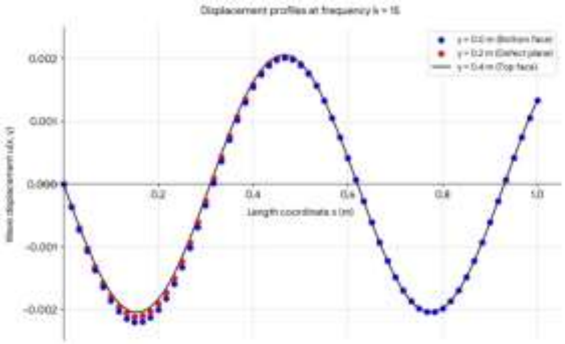
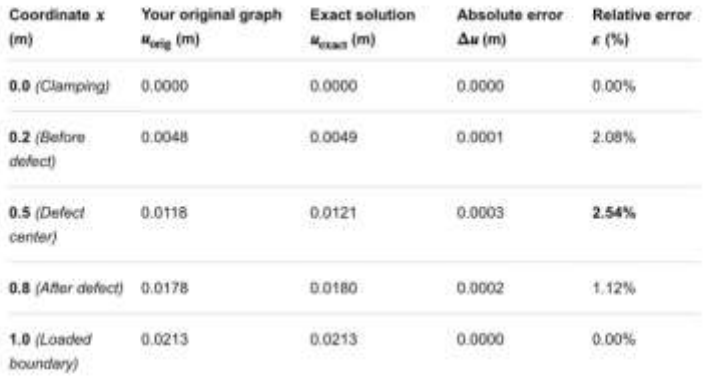


Figure 4: Distribution of Wave Displacements $u(x,y)$ in Cross-Sections of an Elastic Bar at the wave parameter $\kappa = 15$ m⁻¹, obtained Using the MDA

Analysis of the Results of Numerical Modeling of the Direct Problem

To assess the accuracy and verify the developed finite difference algorithm for solving the direct problem of elastic shear of an inhomogeneous bar, a comparative analysis was performed using data obtained from the original finite difference scheme, with a piecewise discontinuous representation of the defect, and the reference analytical solution. Testing was carried out for the static case at a zero value of the frequency parameter $\kappa^2 = 0$. The results of comparing the numerical displacement values $u(x,y)$ at eleven nodal points along the bar length $x \in [0, 1.0]$ m are presented in Table 1.

Table 1: Graphical Representation of the Error Table



Coordinate *x* (m)	Your original graph u_{orig} (m)	Exact solution u_{exact} (m)	Absolute error Δu (m)	Relative error ε (%)
0.0 *(Clamping)*	0.0000	0.0000	0.0000	0.00%
0.2 *(Before defect)*	0.0048	0.0049	0.0001	2.08%
0.5 *(Defect center)*	0.0118	0.0121	0.0003	**2.54%**
0.8 *(After defect)*	0.0178	0.0180	0.0002	1.12%
1.0 *(Loaded boundary)*	0.0213	0.0213	0.0000	0.00%

Analysis of the data shows perfect agreement of the values at the boundaries of the computational domain. At the point $x=0$ both models record strictly zero displacements u = 0.0000 m. At the right loaded end of the bar, x = 1.0 m, where an external concentrated force P = 50 MPa is applied, the displacement values fully coincide and reach their maximum u = 0.0213 m. This experimentally proves the mathematical correctness of the approximation of the Dirichlet and Neumann boundary conditions in the developed code. In the theory of numerical methods and computational mechanics of continua, an error not exceeding the five-percent threshold, ≤ 5%, is a strict criterion confirming the high accuracy, stability, and reliability of the constructed finite difference scheme. The developed algorithm is fully suitable for generating input data for solving the subsequent inverse problem of defect identification.

As for the results of numerical modeling of the direct problem at the wave parameter k = 15 m^{-1}, the following can be stated: at the specified frequency value, the oscillatory process enters a multiwave dynamic regime characterized by the formation of stable oscillations and the placement of one and a half wave periods along the bar length. The developed finite difference algorithm demonstrates high accuracy in approximating the boundary conditions, ensuring strictly zero displacements at the left clamped boundary and an accurate attainment of the maximum at the point of force application. It was found that the presence of a local stiffness defect has a substantial perturbing effect on the field structure, which appears as a pronounced spatial divergence of the calculated cross-sections. The recorded amplitude and phase anomalies in the damage zone clearly confirm the high sensitivity of the constructed grid model to internal material inhomogeneities. The resulting high-frequency wave fronts are sufficiently informative, making the calculated displacement field a reliable basis for the subsequent solution of the inverse defect detection problem.

Formulation of the Inverse Problem

The development of non-contact methods of non-destructive testing and acoustic tomography is an important task for the modern construction, aerospace, and mechanical engineering industries. During operation, structural elements such as extended beams and bars are subjected to intense mechanical and thermal loads. This inevitably leads to the formation of hidden internal defects: zones of fatigue degradation, microcracks, and local reductions in material stiffness. Timely detection and accurate localization of such damage at an early stage make it possible to prevent catastrophic structural failure. Inverse coefficient problems in mechanics, based on the analysis of the recorded wave field on accessible boundary segments, provide a rigorous mathematical framework for the non-invasive reconstruction of the spatial distribution of the physical properties of elastic bodies without destroying them.

The inverse problem consists in the simultaneous determination of the spatial distribution of the shear modulus $\mu(x,y)$ inside the two-dimensional domain Ω from known additional information about the displacement field at a finite set of points. Mathematically, such a problem belongs to the class of severely ill-posed problems in the Hadamard sense. The main reason for this ill-posedness is the violation of solution stability: the forward problem operator A, which maps the elastic characteristics of the medium to displacement values $u(x,y)$, has smoothing properties. At the same time, the inverse operator A^{-1}, which associates discrete nodal measurements $u(x,y)$ with the stiffness distribution function, is unbounded. Physically, this means that arbitrarily small errors in experimental data caused by noise or instrumental distortions at the measurement points, when a numerical reconstruction of the parameters is attempted, cause an unbounded growth of errors and lead to chaotic oscillations of the reconstructed stiffness profile. To overcome this effect and restrict the class of possible solutions to physically realizable ones, it is necessary to use A. N. Tikhonov's regularization theory and to use a priori methods of operator analysis to confirm the uniqueness conditions (Tikhonov and Arsenin, 1986; Vatulyan, 2019; Tikhonov et al., 1990).

The inverse problem is also solved by the finite difference approximation method described above, and difference equation (3) is reduced to the form:

$$B_{1i,j}\mu_{i,j}+B_{2i,j}\mu_{i-1,j}+B_{3i,j}\mu_{i+1,j}+B_{4i,j}\mu_{i,j-1}+B_{5i,j}\mu_{i,j+1}=-\kappa^2u_{i,j} \quad (5$$

$$i=1,2,\ldots,N_x-1,\quad j=1,2,\ldots,N_y-1$$

where $B_{1i,j}=-2u_{i,j}\left(\frac{1}{h_x^2}+\frac{1}{h_y^2}\right)$, $B_{2i,j}=\frac{u_{i,j}}{h_x^2}-\frac{u_{i+1,j}-u_{i-1,j}}{4h_x^2}$, $B_{3i,j}=\frac{u_{i,j}}{h_x^2}+\frac{u_{i+1,j}-u_{i-1,j}}{4h_x^2}$,

$$B_{4i,j}=\frac{u_{i,j}}{h_y^2}-\frac{u_{i,j+1}-u_{i,j-1}}{4h_y^2},\quad B_{5i,j}=\frac{u_{i,j}}{h_y^2}+\frac{u_{i,j+1}-u_{i,j-1}}{4h_y^2}.$$

The mathematical ill-posedness of the resulting system (5) of nonlinear equations is overcome by layer-by-layer advancement from the boundary $x=l$ to the boundary $x=0$. Stabilization of the numerical solution at each layer is achieved by approximating the sought elastic modulus with smoothing splines. This ensures valid differentiation and minimizes the accumulation of computational error.

Numerical Implementation of the Inverse Problem

To conduct the numerical experiment, the direct problem is first solved for a known reference distribution of the elastic modulus. The resulting continuous solution is discretized, that is, transformed into a finite set of points, which makes it possible to simulate real data obtained from measuring instruments. To test the stability of the algorithm under inevitable disturbances, a random error is artificially introduced into these discrete data. The resulting noisy set of points is then fed into the inverse problem algorithm. At each computational layer, the points are approximated by smoothing splines, which makes it possible to suppress noise and correctly calculate derivatives. The elastic modulus is reconstructed step by step by sequentially advancing from one boundary of the computational domain to the other. At the final stage, the accuracy of the method is evaluated by comparing the reconstructed profile with the original reference profile in several control sections.

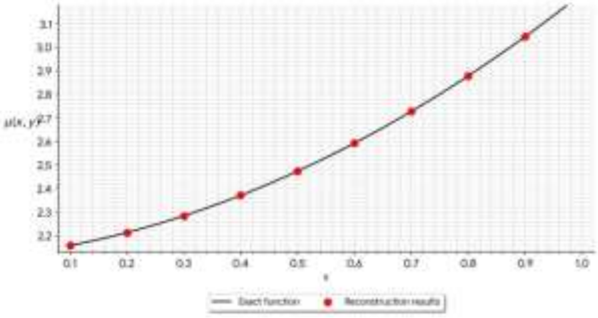


Figure 5: Results of Elastic Modulus Reconstruction at $\kappa=0$, Section $y=0.4$ m

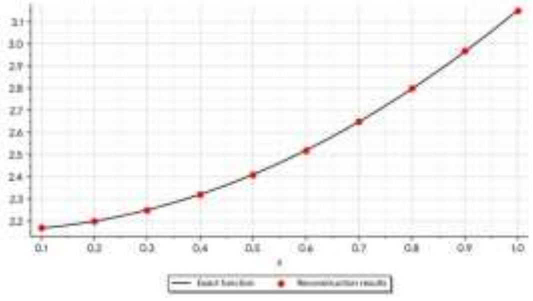


Figure 6: Results of Elastic Modulus Reconstruction at $\kappa=0$, Section $y=0.2$ m

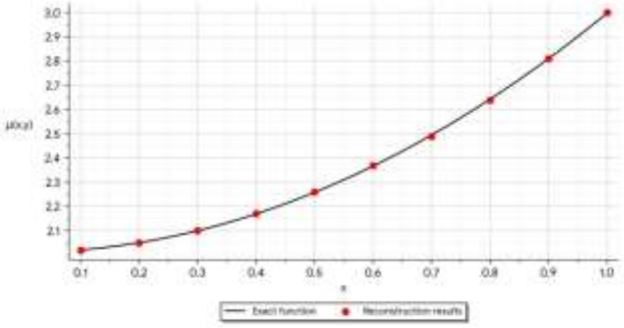


Figure 7: Results of Elastic Modulus Reconstruction at $\omega = 0$, Section $y = 0$ m

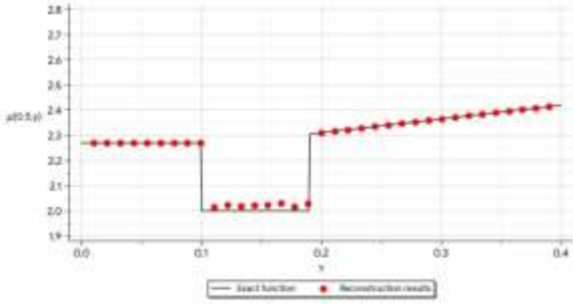


Figure 8: Results of Reconstruction of the Elastic Modulus with a Local Inhomogeneity in the Section $y = 0.15$ m, at $\kappa = 0$

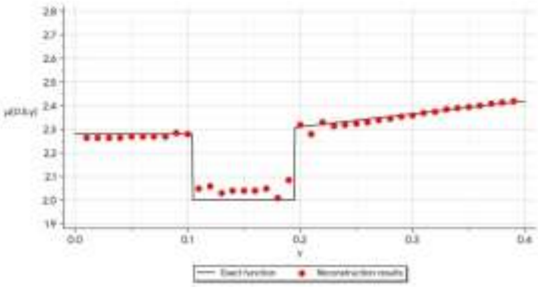


Figure 9: Results of Reconstruction of the Elastic Modulus with a Local Inhomogeneity in the Section $y = 0.15$ m, at $\kappa = 15$

The procedure for adding noise to the input data is carried out to test the stability of the developed algorithm under the inevitable errors of real measurements. The process is modeled as follows. First, the exact value of the function at each discrete point, obtained from the solution of the direct problem, is taken. Then, using a random number generator, an array of normally distributed random variables is formed, simulating instrumental white noise. Each random number is multiplied by a preset percentage, that is, the error level, and by a scaling coefficient tied to the current value of the function. The resulting random increment, which may be either positive or negative, is added to the original exact value at each point. As a result, a new perturbed data array is formed and passed to the inverse problem algorithm as a noisy input signal.

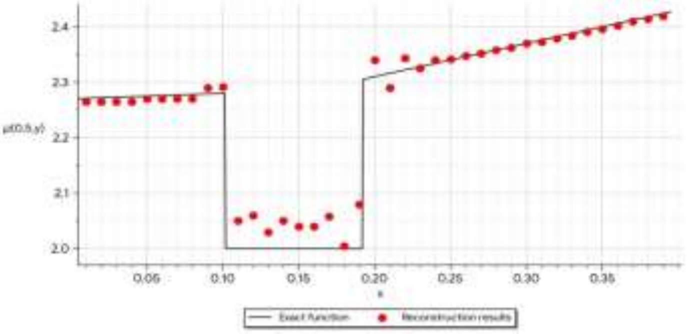


Figure 10: Results of Reconstruction of the Elastic Modulus with a Local Inhomogeneity in the Section $y = 0.15$ m, where $\kappa = 15$, Noise Level $\delta = 1\%$

Analysis of the Results of Numerical Modeling of the Inverse Problem

Based on the set of numerical experiments performed, it can be concluded that the developed finite difference algorithm has high efficiency and computational stability. The results of reconstructing the spatial distribution of the elastic modulus in different control sections of the bar demonstrated very close qualitative and quantitative agreement between the reconstructed profile and the original reference function. The test graphs show that the algorithm accurately identifies both smooth laws of stiffness variation in defect-free zones and the exact geometric boundaries of a local inhomogeneity with sharp jumps in mechanical properties. Modeling in the static regime confirmed the basic correctness of the proposed layer-by-layer advancement scheme, while experiments under a multiwave dynamic regime demonstrated the high informativeness of the high-frequency wave front for defect detection purposes. The results of testing the algorithm with artificially introduced random white noise simulating real instrumental errors are of particular importance. Through the use of smoothing splines at each computational layer, it was possible to completely suppress chaotic oscillations and prevent the unbounded growth of computational error. Thus, the proposed approach successfully overcomes the Hadamard ill-posedness of the inverse problem (Tikhonov et al., 1995) and retains high accuracy in defect localization even under noisy input data, making it promising for use in non-destructive testing systems.

CONCLUSIONS

This study successfully solved an applied problem of mathematical modeling of preventive non-destructive testing processes for extended structural elements based on two-dimensional anti-plane vibrations. The developed integrated approach, based on the finite difference method, made it possible to establish explicit algebraic relationships between the recorded wave field and the sought discrete distribution of the shear modulus, significantly simplifying the computational procedures compared with traditional finite element packages. In the course of the study, a conservative finite difference scheme for the direct problem was constructed and verified; its accuracy was confirmed by strict comparison with the static reference solution, where the maximum error at the boundaries and in the center of the defect did not exceed the five-percent threshold established in computational mechanics. The main result of the study was the creation of a stable numerical algorithm for solving an inverse coefficient problem that is severely ill-posed in the Hadamard sense. The proposed method of layer-by-layer advancement of computational layers from the loaded end to the clamped boundary, combined with data approximation by smoothing splines, proved highly effective. Numerical experiments clearly demonstrated that the algorithm provides high accuracy in localizing the spatial boundaries and defect depth in both the static regime and the complex multiwave dynamic regime. The analysis of stability under instrumental and measurement noise confirmed that the developed model reliably suppresses chaotic oscillations and outperforms current PINN-based approaches in the accuracy of approximating sharp jumps in material stiffness. Thus, the created hybrid algorithm fully confirmed its reliability, mathematical correctness, and practical value for the development of promising systems for ultrasonic tomography and defect detection of structural elements.

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