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RESEARCH ARTICLE

Grade Five Students Common Errors and Misconceptions in Multiplication and Division for Whole Numbers

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ABSTRACT

This study is concerned with Saudi Grade Five students‟ common errors and misconceptions of multiplication and division in whole numbers. It has considered students‟ understanding of multiplication and division concepts and properties. Students‟ errors in multiplying and dividing by multiples of 10, and mental strategies and calculations, and standard algorithms are investigated. The study used both qualitative and quantitative methods. A written test and semi structured interviews were used to collect the data. The target population of the written test was 96 students in four urban schools. Only 10 students of the sample were chosen to take part in interviews. Selected data of the test and the interviews were used in the study in order to address the research questions. The study findings show that some students hold common misconceptions of multiplication and division. There are various common errors in understanding multiplication and division symbols and properties. Furthermore, mental strategies are not used by the majority of students. Finally, there are common errors in the use of standard algorithms of multiplication and division with a lack of understanding of algorithmic procedures.

INTRODUCTION

The teaching and learning of mathematics at the primary level is an essential part of developing a child understands of life. At this level, children are taught numbers and operations, such as multiplication and division. It is crucial for a teacher to teach the operations of multiplication and division in such a way that they support children’s mathematical understanding. One of the most important aspects of teaching and learning multiplication and division is to understand how children comprehend these concepts, and further, how they use these operations to solve real life problems. This important aspect of teaching and learning poses one difficulty, specifically, ‘how do you measure the level of child’s understanding?’ Furthermore, it is possible that when children learn multiplication and division, they make various errors due to a lack of understanding and due to careless mistakes. Moreover, they may have misconceptions about multiplication, such as “multiplication always makes the number bigger”, and in relation to division, “it always makes the number smaller”.

Exploring the multiplication and division errors and misconceptions at the primary level of education may help to raise the standard of students’ mathematical performance. This kind of research has many implications for how teachers teach mathematics, how students learn mathematics, and how mathematics curricula are designed and so on. For instance, students’ errors and misconceptions play a positive role in the teaching and learning of mathematics in terms of providing useful insights into students' thinking and mathematical understanding. These errors and misconceptions can highlight what the difficult mathematical concepts and meanings are; those that require more explanation to be understood, and those mathematical skills that need more practice before they can be applied without mistakes. Thus, mathematics teachers might change their pedagogical strategies in classrooms. Therefore, this study aims to investigate the common errors and misconceptions in multiplication and division of whole numbers for 11-year old Saudi grade five students.

LITERATURE REVIEW

Models of understanding multiplication and division

When students learn about multiplication and division concepts and develop strategies for solving multiplication and division sums and real world problems, they will have already learned about addition and subtraction operations and they will relate those new concepts of multiplication and division to what they already know. This is what called the conceptual structure which is critical in terms of teaching and learning mathematics in primary level. According to Ernest (2000, p.5), 'a conceptual structure is a set of concepts and linking relationships between them'. One of the crucial reasons for teaching and learning multiplication and division is the conceptual development of students' mathematical knowledge.

NCTM (2000, p.33) stresses that students need to be aware of understanding numbers, ways of representing numbers, relationships among numbers, and number systems. For example, to simplify the fraction 210/70, students might use the skill of dividing by 10, then use the fact that $21 \div 7 = 3$, which might be based on one of division's concepts regarding how many times 7 can be subtracted from 21. The understanding of multiplication and division operations develops students' ability to use them for many social practices such as shopping, money, and measurements.

Many studies about students' understanding of multiplication and division found that repeated addition and repeated subtraction are the common models and strategies of multiplication and division. Nunes and Bryant (1996) suggest that a commonly held view of multiplication and division is that they are simply 'different arithmetic operations ... taught after [the pupils] have learned addition and subtraction' (p.144). However, they emphasise that 'it would be wrong to treat multiplication as just another, rather complicated, form of addition, or division as just another form of subtraction. The reason for this is that there is much more to understanding multiplication and division than computing sums' (p.144).

Many studies report that students usually use various strategies to solve multiplication and division word problems, and they acquire different models of multiplication and division (Kouba, 1989; Greer, 1992; Fischbein et al., 1985; Steffe, 1994, Mulligan and Mitchelmore. 1997). However, these various models should be emphasised in the teaching and learning of multiplication and division. Vest (1971) indicates that "since the concepts of whole numbers and their operations are fundamental to the mathematical curriculum, and these are taught with the aid of models, then knowledge of the domain of models is also fundamental" (p.220).

Multiplication models	Division models
Repeated addition	Partitive division (sharing)
The array model of multiplication	The quotative division (grouping)

Table 1: Multiplication and division models

Mental multiplication and division

Mental calculations are essential in everyday uses of mathematics; therefore, teaching and learning mental strategies becomes very important in primary mathematics classrooms. There are advantages to expecting students to use mental calculation methods, such as a stronger 'number sense', a better understanding of place value and more confidence with numbers and the number system (QCA, 1999). Many mental strategies can be used by students to solve multiplication and division sums rapidly.

There are different strategies of mental calculations such as multiplying and dividing by multiples of 10, doubling and halving. For example, a student might prefer to solve the sum 12×14 as (10×14 + 2×14) rather than (8×14 + 4×14) because he or she might know the power of multiplying by 10. In terms of doubling and halving method, Haylock (2006) explains this method as the constant-ratio method for division, which occurs simplifying a division calculation by multiplying or dividing the dividend and the divisor by the same number, thus, the ratio of dividend to divisor does not change. In following Table 2, there are different strategies of mental calculations of multiplication and division:

Mental strategies of multiplication	Mental strategies of division
Recalled number facts	Recalled number facts
Derived multiplicative facts	Division laws
Multiplication laws	
Fractional distribution	

Written methods for multiplication and division

Students solve multiplication and division problems using efficient written methods which mean those methods that are the formal strategies used to solve multiplication and division using pencil and paper and following specific procedures. These mathematical methods have been called standard written algorithms. Students are taught multiplication and division standard algorithms where more difficult calculations are introduced, such as dividing a three-digit number by a two-digit number.

The notion of using errors and misconceptions in teaching mathematics

There is no doubt that students' learning of mathematics is not an accurate process; it would be very unusual for students to undergo this process without making mistakes and misunderstanding some mathematical facts and concepts because each individual might construct his or her own knowledge in a different way. As students construct their own meanings, it is inevitable that they will make errors (Hansen et al. 2005). However, it is necessary to clarify the meaning of a mistake or an error and a misconception. Koshy (2000, p.172) says that 'a mistake is described as a wrong idea or wrong action; a misconception is defined as misunderstanding'. Spooner (2002) defines a misconception as the product of a lack of understanding or the misapplication of a rule or mathematical generalisation whereas an error can be the result of a misconception, but could also be caused by a number of other factors, such as carelessness, problems in reading or interpreting a question, and a lack of number knowledge. Thus, if a student has a misconception of such a mathematical concept, she or he might make mistakes.

A consideration of how students acquire mathematical knowledge should include the identification of the reasons behind such misconceptions and why students make common errors. Teaching mathematics in a way that avoids creating misconceptions is impossible and many misconceptions will remain hidden unless teachers make specific efforts to uncover them (Anghileri 2000). However, some misconceptions and mathematical errors can be avoided by teachers' awareness, an effective choice of tasks and activities, and more explanation and discussion with students.

Errors and misconceptions in multiplication and division

Studies have reported that students' achievement in solving multiplication and division word problems is influenced by misconceptions such as "multiplication always makes a number bigger,", "division always makes a number smaller," and "you cannot divide a smaller number by a larger number" (Bell, Fischbein, and Greer, 1984; Brown, 1981; Fischbein et al., 1985). Graeber and Tirosh (1990) report that students in the fourth and fifth grades hold some of these conceptions, which may interfere with their work with decimals. However, an accurate understandings of the ideas students have about the multiplication and division of whole numbers might be useful in building understandings of these operations with decimals and of further mathematical concepts such as fractions and percentage .One of the reasons for these misconceptions, such as 'multiplication makes bigger' and 'division makes smaller', is that students understand only one concept of multiplication, that is, repeated addition (Greer 1988, Hart 1981). DfES (2005) indicates that there are various errors and misconceptions in multiplication and division at primary level.

THE METHODOLOGY OF THE STUDY

Aims of study

This study aims to contribute to mathematics education research in terms of developing the teaching and learning of mathematics in primary level. Undoubtedly, when students learn mathematics, they have misconceptions and make errors. Thus, the primary aim of this study is to investigate Saudi students' misconceptions and common errors in whole number multiplication and division in Grade Five. Furthermore, the study aims to investigate students' understanding of multiplication and division concepts and how they solve multiplication and division problems or sums in mental and written methods.

Research Questions

Q: What are the common errors and misconceptions in multiplication and division of whole numbers for Saudi grade five students?

Research methods

The study used both qualitative and quantitative methods to find the answers to the research questions. There was a test, which was divided into three parts: understanding multiplication and division, mental strategies, and written methods. The test was designed to collect students' errors in multiplication and division of whole numbers. The test was not a satisfactory method to address students' mathematical understanding. Hence, there was a need to use semi-structured interviews (Hunting, 1997, Ginsburg, 1981) because one of the considered purposes of the interview was to collect in-depth information to achieve some of the purposes of this study. The interview focused on the multiplication and division concepts, and how students understand them.

The sample

This study focused on Saudi Grade Five students aged between 11 and 12 years old. The number of participants was 96 students in four urban schools. The sample of students was only boys. The schools included different types; there were two state schools (54 students) and two private schools (42 students). The four primary schools were selected by opportunistic sampling (Cohn et al, 2007) and from each school one class was selected randomly. Then, after students' answers had been collected, 10 students were chosen to take part in semi-structured interviews. The 10 interviewees were chosen regarding their answers of the written test.

ANALYSIS OF THE RESULTS

Analysis of the written test

Table 3 illustrates the students' responses to four items about multiplying and dividing of multipliers of tens and their common errors: 53% of students calculated 28×100 (T1) incorrectly, with two common errors occurring, 128 and 280 at 37% and 18% respectively; 36% of 96 students were able to answer T2 correctly, whereas 43% of students answered it incorrectly, with 15% giving the answer as 100. The task 206÷10 (T3) had the lowest percentage of students answering correctly at only 3% among the four items, and the highest percentage of incorrect answers at 64%. The most common errors for T3 were 20 and 2060, at 11% and 10% respectively. The last task (T4) in this table was about dividing by a multiplier of 10, which is 30. 41% of students answered it incorrectly, with 3 and 1920 as common errors at 8% each.

Items	Tasks	Response pattern			Common errors	
		Correct answers	Incorrect answers	No answers	Errors	Frequency
T1	28 × 100 = □	43%	53%	4%	128	37%
					280	18%
T2	54 × □ = 540	36%	43%	21%	100	15%
					50	10%
T3	206 ÷ 10 = □	3%	64%	33%	2060	10%
					20	11%
T4	240 ÷ □ = 8	13%	41%	46%	1920	8%
					3	8%
					192	5%

Table 3: The percentages of students' responses for multiplying and dividing of multipliers of tens, and common errors.

Division-based multiplications

Students' responses to two division-based multiplications are illustrated in table 4. It can clearly be seen that approximately 10% of 96 students were able to answer T5 and T6 correctly. However, the majority of students were not able to solve the items correctly. Three quarters of the students answered T5 incorrectly and 64% of them calculated the answer incorrectly as 3. The majority of incorrect answers (72%) for T6 were given as either 7 or 6, at 49% and 12% respectively.

Items	Tasks	Response pattern	Common errors

		Correct answers %	Incorrect answers %	No answer %	Errors	Frequency
T5	$\square \div 24 = 8$	10%	75%	15%	3	64%
T6	$\square \div 5 = 35$	9%	72%	19%	7	49%
					6	12%
					40	4%

Table 4: The percentages of students' responses to division-based multiplications, and common errors.

Understanding divisions with remainder and repeated addition concept of multiplication

Table 5 shows students' responses to two tasks about their understanding of division with the remainder and repeated addition concept of multiplication. There were three common mistakes among 58% incorrect answers to task T7: 18% of incorrect answers were given as 45, and the number 1 was given as an incorrect answer by 14%. Task T8 was correctly answered by 3% of 96 students, whereas 39% of students were not able to solve it. There were two common errors for T8, namely 1 and 4736, at 27% and 14% respectively.

Items	Tasks	Response pattern			Common errors	
		Correct answers %	Incorrect answers %	No answer %	Errors	Frequency
T7	$\square \div 9 = 5$ remainder 6	18%	58%	24%	45	18%
					1	14%
					20	10%
T8	$4735 \times 2814 = 13324290$ $4736 \times 2814 = 13324290 + \square$	3%	39%	58%	1	27%
					4736	14%

Table 5: The percentages of students' responses to division with remainder and repeated addition concept of multiplication tasks.

The mental written test

Analysis of the multiplication and division mental test included looking at students' correct answers and the strategies used to solve mental tasks. In addition, the author analysed the common errors of the incorrect answers for the multiplication and division mental tasks.

Mental multiplication

As shown in table (6), 41 students were able to answer the task 61×4 using four strategies; although 16 students did not show their solution strategy. Almost half of the correct answers were calculated using standard algorithms as common strategy. Only 2 students used distributive law (4×1+ 4×60 = 4+240 = 244), and 2 students used counting up (61+61+61+61= 244). Another 2 students used doubling (61+61= 122+122= 244). It is clear from the table that 6 of 54 students who answered incorrectly gave 64 as an answer. Only 14 of 96 students correctly answered the task 11×17 by using standard algorithms (8 students) and additive distributive law (2 students, as 11×17= 10×17 + 1×17 = 170+ 17= 187). However, 22 of 70 students who were incorrect gave 17 as a common mistake. It is noticeable that the task 25×32 was the most difficult mental multiplication task and only 8 students were able to answer the task correctly. Among 66 incorrect answers there were two common errors, namely 70 and 610, given by 11 students each.

Items	Tasks	The number of correct answers	Strategies used for correct answers		The number of incorrect answers	Common errors
			No	Type of strategy		
M1	61×4	41	19	Standard algorithms	54	64 (6 times)
			16	No explanation		
			2	Additive distributive law		241 (5 times)
			2	Counting up		
			2	Doubling		
M2	11×17	14	8	Standard algorithms	70	17 (22 times)
			4	No explanation		
			2	Additive distributive law		107 (5 times)
M3	25×32	8	4	Standard algorithms	66	70 (11 times)
			2	Distributive law		
			2	No explanation		610 (11 times)

Table 6: Students' responses and common errors for mental multiplication problems.

Mental division

There were two mental division tasks in the mental written test as shown in table 7. The first task was 618÷3, which was correctly answered by only 12 students out of 96. Half of the correct answers used an additive distributive law strategy (618÷3 = 600÷3 + 18÷ 3 =200+6 = 206), while the other half of students who answered correctly did not write the strategy they used. The number 26 was a common mistake among 54 incorrect answers, given 13 times. The task 875÷25 was the second mental division task and was answered correctly by 9 students; 4 of these 9 students used a distributive strategy (875÷25= 800÷25 + 75÷25 = 32 + 3 = 35) and only 2 students used the fact 25×4 = 100 (each 100 is 4 twenty-five, 800 is 8×4= 32 twenty- five plus three 25 in 75, the answer is 35).

Items	Tasks	The number	Strategies used for correct answers	The number	Common errors

		of correct answers	N	Type of strategy	of incorrect answers	
D1	618÷3	12	6	Distributive law	54	26 (13 times)
			6	No explanation		
D2	875÷25	9	4	Distributive law	54	No common errors
			2	Based on the fact 25×4= 100		
			1	Multiplication table of 25		
			1	Standard algorithms		
			1	No explanation		

Table 7: Students' responses and common errors for mental division problems.

The written problems of multiplication and division

In the analysis of standard algorithms, students' incorrect solutions are classified into three categories. The author has coded the answer as 'A' where a student generally understands algorithms with some errors in implementing them, such as 'incorrect multiplication facts'. The second category, which coded as N, is for a student who misunderstands one step of the procedure or has difficulty in doing it. Finally, when a student does not understand algorithms and does not know what they are supposed to do, writing anything without any procedure to solve a long multiplication and division problem, the answer is coded as 'NA'. The following figure 1 shows an example of each category:

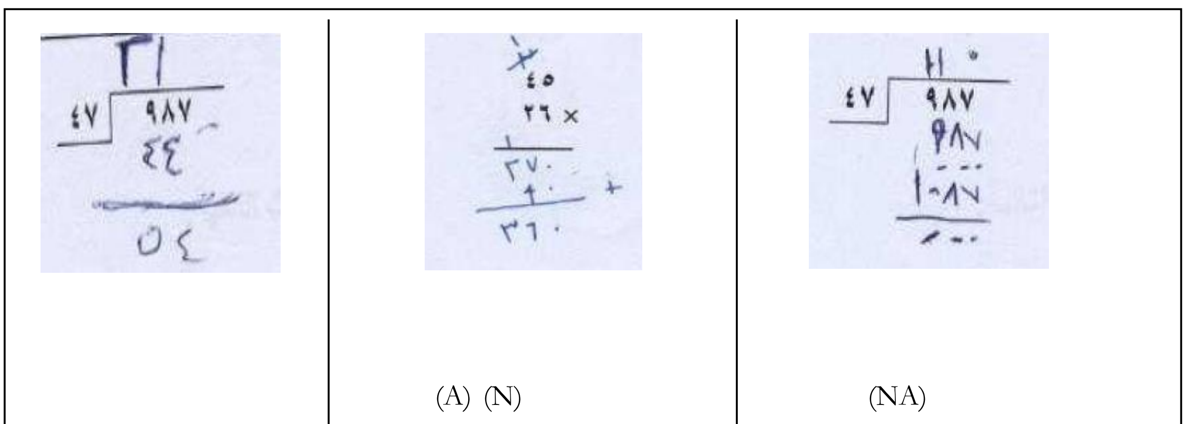


Figure 1: Examples of each category of students' incorrect standard algorithms.

Multiplication problems

Table 8 below shows the percentage of students' answers to three standard multiplication algorithms and their common errors. The difficulty of each problem increases as the students' progress from item M1 to item M3 as shown in figure 2. This is reflected by the percentage of correct answers to each problem: 43% of students who answered incorrectly the task 27×8 we're multiplying units only. With M2 and M3 the common error was multiplying vertically only (e.g. 45×26= 110), at 20% and 17% respectively. It is interesting to note that items M1 and M3 have either 7, 8 or 9 as the final digit. A common mistake with these items was incorrect multiplication facts. There is one type of error which applies to all the incorrect answers of all items, namely an incorrect algorithm procedure.

Items	Format	Response pattern			Errors* %	Type of errors
		Correct answers %	Incorrect answers %	No answer %		
M1	27×8	52	47	1	43%	Multiplying units only
					28%	Incorrect multiplication facts
M2	45×26	23	72	5	34%	Incorrect algorithm procedure
					20%	Multiplying vertically only
					14%	Not putting zero
M3	318×29	11	69	20	28%	Incorrect multiplication facts (8×9)
					27%	Incorrect algorithm procedure
					17%	Multiplying vertically only

Table 8: Students' responses and common errors for standard multiplication algorithm problems.

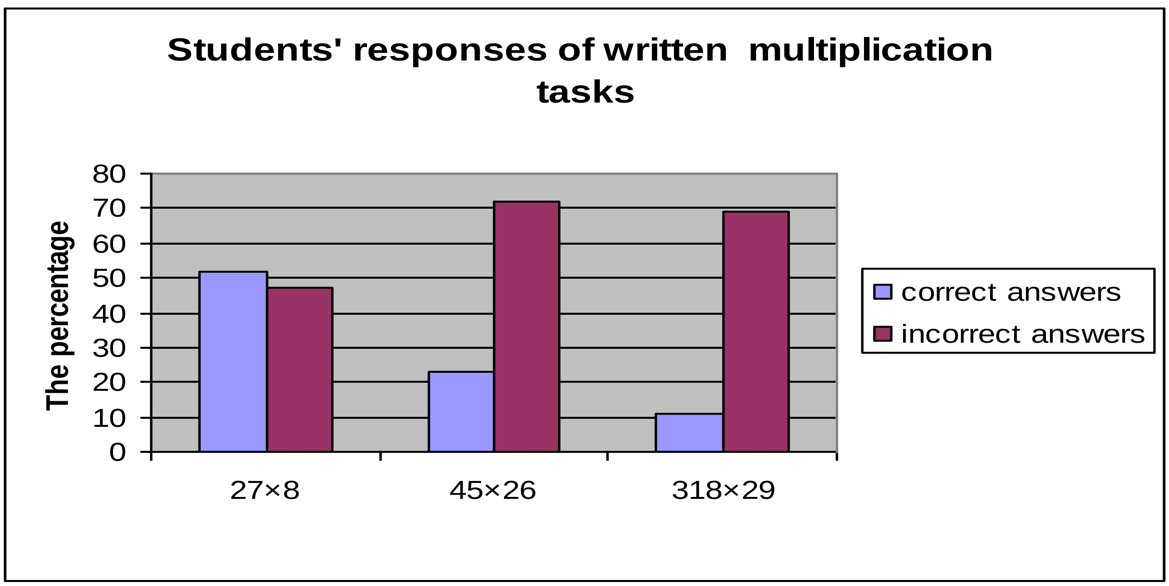


Figure 2: Students' responses to written standard algorithms for multiplication tasks.

Division problems

There were four division tasks to be answered using the standard written algorithms. It can be seen in table 9 that no more than 30% of 96 students were able to answer all four division tasks. Surprisingly, the task 987÷47 (D4) had the lowest proportion of incorrect answers (40%), whereas the task 824÷4 (D1) had the highest proportion at 70%. Task D1 had a common error, with 43% giving the incorrect answer of 26. The two tasks 158÷7 (D2) and 672÷8 (D3) had the same percentage of incorrect answers, at 48%. In task D2, 22% of incorrect answers were incomplete by finding the remainder. The same percentage applied to task D3, with incomplete answers and wrong multiplication facts. The highest percentage of student errors for all four division tasks was an incorrect algorithm procedure, from 39% to 61%.

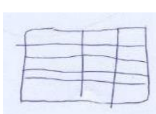
Items	Format	Response pattern			Errors %	Type of errors
		Correct answers %	Incorrect answers %	No answers %		
D1	824 ÷ 4	13	70	17	26: 43%	A Common error
					57%	Incorrect algorithm procedure
D2	158 ÷ 7	31	48	21	61%	Incorrect algorithm procedure
					22%	Not completed by finding the remainder
D3	672 ÷ 8	24	48	28	39%	Incorrect algorithm procedure
					39%	The step of dividing 67 by 8
					22%	Others include multiplication facts-uncompleted answers
D4	987 ÷ 47	29	40	31	45%	Incorrect algorithm procedure
					34%	The step of dividing 98 by 47
					21%	Others

Table 9: Students' responses and common errors for standard division algorithms problems.

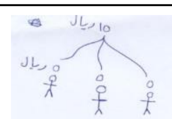
ANALYSIS OF THE INTERVIEW

The first part: definition and properties of multiplication and division

All 10 interviewees were asked about the meaning of the multiplication and division symbols before they were asked to draw pictures showing 5×3 and 15÷3. All students reported that the meaning of multiplication is repeated addition; 7 students interpreted 5×3 as 5 added 3 times, whereas 3 out of 10 students interpreted 5×3 as 5 added 3 times or 3 added 5 times. On the other hand, 3 students indicated that the meaning of division is inverse operation of multiplication. However, the majority of interviewees defined the concept of division as sharing. For example, one student interpreted 15÷3 as "you have 15 riyals, and you want to share it between 3 people, so each one takes 5 riyals". Students' drawing representations of 5×3 and 15÷3 are illustrated in the following figure 3:

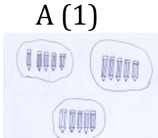


A (1)

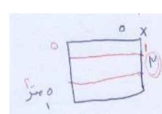


B (1)

B (2)



A (2)



B (3)

Multiplication pictures

Division pictures

Figure 3: Students' drawing representations of 5×3 and 15÷3.

There were 8 students out of 10 who drew three groups of five items to show 5×3 (e.g. picture A1), and only 2 students out of 10 who drew it in columns and rows (e.g. picture A2). Students' pictures of the division sum 15÷3 are divided into three categories. Firstly, 8 out of 10 students drew pictures of fifteen items divided into three parts (e.g. picture b1). Secondly, a student drew a cake divided into 15 pieces to be shared by 3 boys (picture b2). The last picture (b3) is what a student drew to represent 15÷3. It is a piece of land which has an area of 15 square meters. This is divided into three $5m^2$ pieces of land to be shared by 3 people.

In the question about commutative law of multiplication, 16×17 = 17 ×? All the interviewees answered correctly, giving the missing number rapidly within 4 seconds. There were 6 interviewees who immediately indicated that the multiplication is a commutative operation, 3 interviewees justified their answers by saying that 16×17 and 17×16 have the same answer, and 1 student justified his answer by incorrectly stating that it was because 16 is multiplied by 17 on the other side.

Only 2 students out of 10 gave 18 as the correct answer to the question asking if 25×18 is more than 24×18, with the multiplication concept being based on the idea of repeated addition. However, the majority of students said that the difference between 25×18 and 24×18 is 1. The following example shows the reasoning behind the incorrect answer given by the majority of students:

I: Can you tell me how much more 25×18 is to 24×18.

S: 25×18 is more by 1.

I: Why?

S: Because 24 is less than 25 by 1.

Furthermore, 5 students were able to narrate a story or a word problem about division with remainder. There were 2 students who gave stories about division without remainders, and 3 out of 10 students were not able to answer this question. All the remainders of the stories given by the 5 students were the same number, which was one.

The second part: misconceptions of multiplication and division

The interview data indicates that there are misconceptions about multiplication and division of whole numbers held by Saudi grade five primary level students; 5 out of 10 students believed that multiplication always results in a number bigger than each number in the multiplication problem. On the other hand, half of the interviewees reported that multiplication does not always make a bigger number, as shown in the following example, which is the response of one of the students:

I: Does multiplication give an answer that is bigger than each number in the question?

S: Yes, 3×6 equal 18.

I: Is that always correct for multiplication?

S: No.

I: Can you give me an example?

S: 1×1 = 1 and 1×9 = 9.

Regarding the students' responses, 4 of the students hold the misconception that division always results in a number smaller than both numbers in the division problem, whereas 6 of the interviewees stated that division does not always make a smaller number, giving various examples. The next example shows one student's misconception in this regard:

I: If we divide two numbers, is the answer smaller than the two numbers?

S: ...Yes.

I: For example?

S: 3÷3= 1.

I: Is that always correct for division?

S: No…yes it is always correct.

In addition, all 10 interviewees believed that it is not possible to divide a small number by a bigger one. The majority of students indicated that the big number should be divided by the small number.

DISCUSSION

Multiplication and division understandings and concepts

The definitions and meanings that the students produced for multiplication symbols and tasks suggest that the main model of multiplication is repeated addition. This understanding of multiplication is clearly indicated by the students in the interviews and illustrated by their pictures of 5×3 (figure 3). Although out of 10 students, two interviewees understood multiplication as repeated addition, they presented 5×3 as an array model of multiplication. From my experience as a mathematics teacher, Saudi mathematics teachers do not place enough emphasis on this model although it is effective for teaching multiplication properties. According to data obtained during this study, the sharing model of division was held by the majority of students. However, three students defined division operations as the inverse operation of multiplication. The low number of students who understood the relationship between multiplication and division indicates that there is a need to teach students how to understand the link between these two operations through exploring this relationship by themselves, for example, encouraging students to derive two division facts from one multiplication fact such as:

24÷6 = 4

4×6 = 24

24÷4 = 6

Additionally, this link can be used for further facts such as 40×6, 4×60, 40×60, and 4×600 and so on. The majority of students' drawings of 15÷3 illustrated the model of partitive division when students divided 15 into three groups. As shown in the figure 4, two figures presented 15 as one unit, which was then divided into three groups. The picture A1, as explained in the analysis chapter, shows how this student used the concept of a rectangle and measured its area to divide a land area (15 m^2) into three small areas of land, of 5 m^2. The answer, which is 5, in this picture is one whole unit because the student might unitise the five as one thing. This student showed by his picture that he understood multiplication, division, and calculation of a rectangular area. The second picture, A2, presents 15 as whole thing (circular cake) which is divided into 15 parts, then between three persons. This kind of understanding of the concept of division is important to facilitate the concept of fractions for students.

A1 A2

Figure 4: Students' pictures of 15÷3 as the land and circular cake.

The study proves that students have some misconceptions regarding the multiplication and division of whole numbers. For example, half of the students in the interviews thought the answer of multiplying two numbers is always bigger than the two numbers. In addition, some students thought that the division of two numbers always gives an answer smaller than either of the two numbers. Students who have these kinds of misconceptions were supposed to be asked why they think as they do, but the investigation did not include this question in the interviews. These misconceptions might be due to too great an emphasis being placed on only one concept for multiplication and for division.

As discussed above, the majority of students thought of multiplication as repeated addition and division as the sharing model. For example, the sharing (partitive) model emphasises some rules, such as the rule that the divisor must be a whole number and the product must be less than the dividend. These rules might be the basic foundation of the misconception ‘division always make smaller’. Moreover, Greer (1988) and Hart (1981) claim that when students understand only one concept of multiplication (repeated addition), this leads to the misconception ‘multiplication always make bigger’. This misconception can be clarified only by performing operations with all kinds of numbers (Nesher, 1992). Although ‘multiplication always make bigger’ and ‘division always make smaller’ are correct concepts with whole numbers, teachers ought to be aware that students might generalise them to use with decimal numbers. Many studies show that students and pre-service teachers have these misconceptions with decimal numbers.

One of the solutions to the problem of such misconceptions, for example, ‘division always makes smaller’, is to introduce the division concept ‘sharing’ with problems such as ‘4 shared between 8’ in practical situations and discussing its contrast with ‘8 shared between 4’ (Fischbein et al., 1985, Anghileri, 2000). Furthermore, although Saudi primary students do not use calculators in learning mathematics at this level, teachers can use calculators to show students there is a difference between entering 8÷4 on a calculator or 4÷8, while there is no difference between entering 8×4 or 4×8. However, I believe that although using calculators is an effective tool for teaching and learning multiplication and division, they should be used carefully and in an appropriate way. For example, students’ understanding of doubling numbers, multipliers of 10, and the inverse relationship between multiplication and division can be developed through activities based on using calculators.

Another misconception, which was held by 12-year-old students in this study, is that dividing a smaller number by a bigger number is impossible. All 10 interviewees in this investigation had this misconception. Most of the students claimed a bigger number should be divided by a smaller number. This misconception was also addressed in the Newstead et al. (1996) study. It seems to me that this misconception is believed by students because they understand the division concept as sharing. For example, one student gave an example to support his thought of dividing by a bigger number:

I: Is it possible to divide a smaller number by a bigger number?

S: No.

I: Can you tell me, why?

S: Because…for example, we cannot divide 5 pens between 10 students.

Two important issues are indicated by the student’s response. First, it is vital to notice how this student mathematised the division concept from the real life. Students need to be encouraged to use this kind of mathematisiation to improve their mathematical learning. Secondly, he represented the dividend as something that cannot be subdivided into smaller quantities. Mathematically, 5÷10 = 0.5 is an acceptable correct solution in abstract symbols, whereas five pens as concretes cannot be divided between 10 students equally. This shows the awareness of using concretes and manipulatives for teaching and learning mathematical concepts and of how to use them to support the understanding of the abstract use of symbols. Moreover, representing the dividend in such division problems by appropriate examples plays a key role in the understanding of division. For example, using the concept of money with a dividend was effective in proving that dividing a smaller number by a bigger number is possible, as shown when one student changed his misconception.

Encouraging students to discuss their understandings and methods with the teacher and other students is important in addressing and clarifying misconceptions. For example, the following discussion with one interviewee indicates that some misconceptions can be clarified through questioning:

I: Is it possible to divide small number by bigger number?

S: No.

I: Can you tell me, why?

S: it is impossible….must be the inverse the big number by small one.

I: Ok, is it possible to divide one Riyal to two students?

S: Yes.

I: How much each one?

S: half Riyal.

I: So, can we divide 1 by 2?

S: Yes.

This kind of discussion, which based on questioning and interacting with those who have some misconceptions, need to be considered by teachers. It could be said that such misconceptions are hold by children because they are not given the completed meaning of a mathematical concept. For instance, if a teacher teaches division by giving only examples about large numbers divided by small numbers, students might think the inverse would be impossible.

Multiplying and dividing by 10 and its multipliers

On the question of 28×100, this study found that answers of 128 and 280 were common errors for this task. The error 128 might be as a result of the thought that multiplying by 100 is the same as multiplying by one with ignorance of the significance of the zeros, or of students internalising the symbol of (×) as (+) and adding the two numbers. On the other hand, the error 280 might due to the generalisation of multiplying by 10 for all multipliers of 10 such as 100 and 1000. This generalisation could be the same reason for the common error 100 for the task 54 × ? = 540, with 15% of incorrect answers. However, the skill of multiplying by 10 and 100 is a powerful skill for understanding multiplication and division as well. For example, when a student understands how to multiply by 10 and 100, he or she can use this understanding for mental and written calculations, such as the 12 in the task 23×12 can be distributed as 10+2 because multiplying by 10 makes the calculation easier. It is somewhat surprising that only 3% of students solved 206÷ 10 with the common error 20, with 11% of incorrect answers. This task includes two important skills: dividing by 10 and finding the remainder. Another common error for this task was 2060, the answer given by 6 students who might have thought that multiplying and dividing by 10 is the same as adding one zero.

The task 240 ÷ ? = 8 had two common errors: 3 and 1920. The error 1920 indicates the lack of understanding of the inverse relationship between multiplication and division. This is because those students who answered with this error did not understand how to find the missing number (the divisor). It is critical the relationship between the dividend, divisor, and the product be understood by students. In addition to these division concepts, students need to understand the concept of division with a remainder. For example, the number 45 was the most common error for the task of

? ÷ 9 = 5 remainder 6. The reason for this error is that the students did not add 6 to the product of 9×5. They might not know the meaning of the remainder and its relationship with the divisor, which should be bigger than the remainder. Parmar (2003) indicates that students at primary level have difficulty with remainders. Related to this, only five out of 10 students were able to give a story about division with a remainder, such as this student:

I: Can you tell me a story or word problem about division with a remainder?

S: ....Students succeeded, for example, there were five students and the teacher had 7 certificates. He distributed the certificates to the students… each one got one and 2 certificates were left.

Interestingly, the remainder of stories given by four students was one. These examples of the remainder indicate that attention should be paid to why those students limited the remainder to only one. This might be because it is easy for students to give a story with only the number one as the remainder or because of a lack of understanding of the range of a remainder from 'zero' remainder to the 'divisor minus one' remainder.

Interpreting multiplication and division symbols and problem formatting

Most students in the interviews interpreted the meaning of the symbol × as 'lots of' and 'multiply' whereas the symbol ÷ was interpreted as 'divided by' and 'shared by'. The majority of students' pictures of 5×3 indicate that they interpreted 5×3 as three sets of five. This then raises the question regarding the extent to which the interpretation of 5×3 as 'three sets of five' or 'five sets of three' is effective to facilitate the understanding of the multiplication symbol. Haylock and Cockburn (2003) say that the strict interpretation of, for example, 9×3 is three sets of nine and it would be appropriate to establish the commutative nature of multiplication (for example, 5×3 = 3×5) before introducing the formal representation in symbols. The study shows that the commutative law of multiplication is comprehended by the majority of 12-year-old Saudi students. However, teachers should be aware of the generalisation of this law to the division concept.

Understanding the various ways of presenting multiplication and division tasks is very important. For example, although the idea of the multiplication task (G8) is based on the early concept of multiplication, which is repeated addition, 97% of students were not able to answer it, with both 39% incorrect answers and 58% no answers.

(G8)

4735	×	2814	=	13324290		
4736	×	2814	=	13324290	+	

The factors leading to this high proportion of incorrect answers might include not only the idea of the task, but also the way of presenting this task. It can be said that the large number in the task makes it difficult for students. However, the same idea was investigated in the question of how much more is 25×18 than 24×18. Still 8 out of 10 interviewees were not able to answer this question. It seems to me that students are not familiar with this link between any two successive multiplication facts, such as 2×3 and 3×3. Thus, students should be taught key knowledge about the meaning of multiplication and division, such as the idea of task G8.

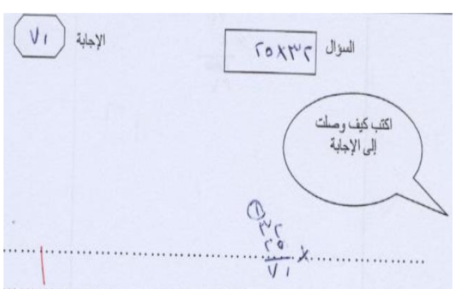
Furthermore, going back to the argument of the presentation of tasks, almost two-thirds of students were not able to answer the task 54 × ? = 540 which was about multiplying by 10. This might be because students usually do tasks in the format 54×10 = ?. Presenting the tasks in various formats is a leading strategy to support students' understanding of the elements of a task involving both missing numbers, such as multiplicand, multiplier, and divisor so on, and the symbols, for example, × , ÷ and = .

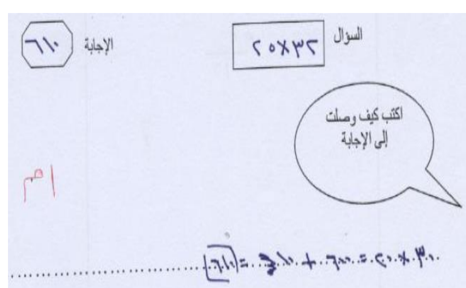
There were common errors for the two tasks ? ÷ 24= 8 and ? ÷ 5= 35, that is, 3 and 7 respectively. As reported in the finding of the data, each error accounted for around three-quarters of the incorrect answers for each task. As reported in Ryan and Williams (2007), 27% of 11-year-old English students gave the same error for the task ? ÷ 5= 35. Their diagnosis of this error includes syntax complexity (working backwards) and division is commutative as a misconception. However, this diagnosis might not be applicable to the another task ? ÷ 24 = 8 at least for the reason of working backward. This type of error could be caused by a student having only one format of division sentence in her or his existing knowledge, namely, that a larger number (the dividend) is divided by a smaller number (the divisor) to calculate the product. For example, when a student sees a task (for example, ? ÷ 24 = 8), he or she then internalises all the numbers and symbols of the task and rearranges them in the same way as in the existing format in his or her mind.

Mental calculation of multiplication and division

The data of this study indicate that there are common errors in multiplication and division mental calculations. Students' answers illustrate the lack of understanding of how to do mental multiplication and division. This might be as a result of the inadequate teaching of mental calculations. There were three multiplication tasks to be solved mentally in this study. Half of the correct answers of the three tasks 61×4, 11×17, and 32×25, were solved using standard algorithms, as illustrated in students' explanation area in the written test. In the UK, Foxman and Beishuizen (2002) reported that there was a tendency by 11-year-old English students in the 1987 APU Survey to use algorithms in mental multiplication questions. The most common mistake for all tasks is that of multiplying the two numbers in the task through units by units and tens by tens. For example, 64 and 17 were common errors of the two tasks 61×4 and 11×17.

The task 32×25 was the most difficult of the mental multiplication questions with only 8 successful answers. Only two students used the distributive law as a mental strategy. There were two common errors, 70 and 610, as shown in the figure 5. Both errors illustrate the misunderstanding of doing mental multiplication by multiplying only units by units and tens by tens. However, the error (A1) 70 shows a further misunderstanding, that is, the lack of understanding of the place value of numbers with the negative effect of standard algorithms. They multiplied 2 by 5 to get 10, put zero down and carry one above 3, then multiplied 3 by 2 to get 6 and add 1 to get 7. This supports the evidence of the negative effect of algorithms in mental calculations, as reported in the study of Heirdsfield et al. (1999) and Kamii et al. (1993).





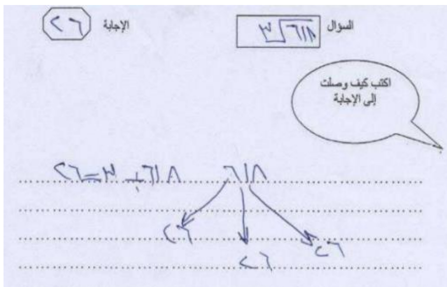
(A1) 32×25 = 70 (A2) 32×25 = 610

Figure 5: two common errors of the mental calculation of multiplication task 32×25.

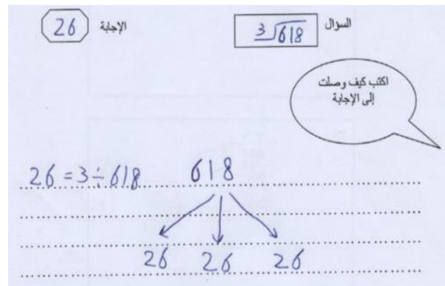
The difficulty of this task could be because it is two numbers multiplied by two numbers, although it might be answered using various mental strategies. For example, students might use the associative property of multiplication and the factors of 32, which are 4 and 8, to solve this task as 25×(4×8) =

(25×4)×8 = 100×8 = 800. Thus, it is important to teach students to use the factors and properties of numbers through various classroom activities. Learning mathematical knowledge should not involve memorization without sense being made of the process and should involve playing with facts to indicate teachers' own high levels of understanding of this knowledge.

Regarding mental division, there were two tasks: 618÷3 and 875÷25. According to the test, only 12 students were able to solve the task 618÷3 correctly and 9 students were able to solve the task 875÷25. The common error 26 for the task 618÷3 as in the figure 6 illustrates that some students have difficulty understanding place value and assessing the reasonableness of their answers. They divide 6 by 2 to get 3, and 18 by 3 to get 6, then write them together.



(The original picture)



(The translated)

Figure 6: the common error of the mental division task 618÷3.

However, students' correct answers show the various mental strategies used by students for mental multiplication and division. The most used strategy by students was the additive distributive law for all tasks. Izsak (2004, p 39) suggests that 'if students develop initial understandings of the distributive property in the context of whole-number multiplication, they will be better prepared to apply the property in other domains such as fractions and algebra'. For example, four out of nine students used distributive law for the task 875÷25. The following figure 7 shows one student's correct answer by this law:

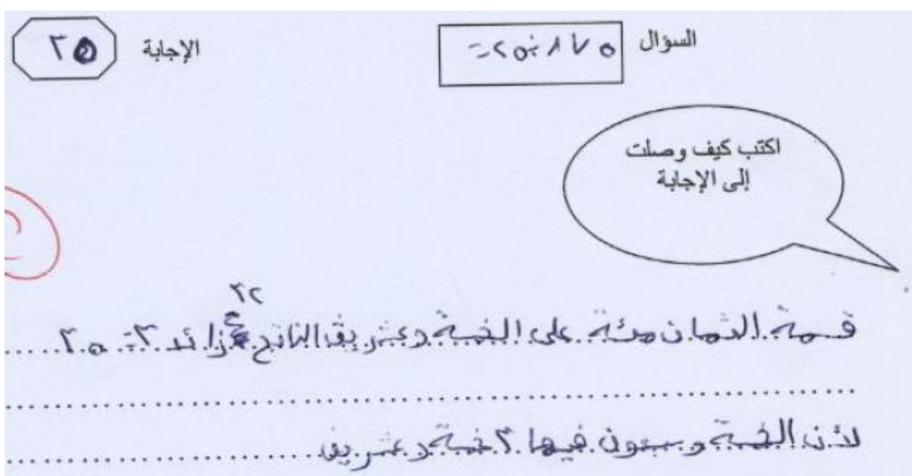


Figure 7: a correct answer of the mental division sum 875÷25 using distributive law.

The student explained his scheme saying, "I divided eight hundreds by twenty-five and the answer is 32 plus 3 equals 35 because seventy-five is 3 twenty-fives." Furthermore, another mental strategy used in multiplication is doubling, which was used by two students for the task 61×4 as illustrated in the following figure 8 as (2×61= 122+122 = 244):

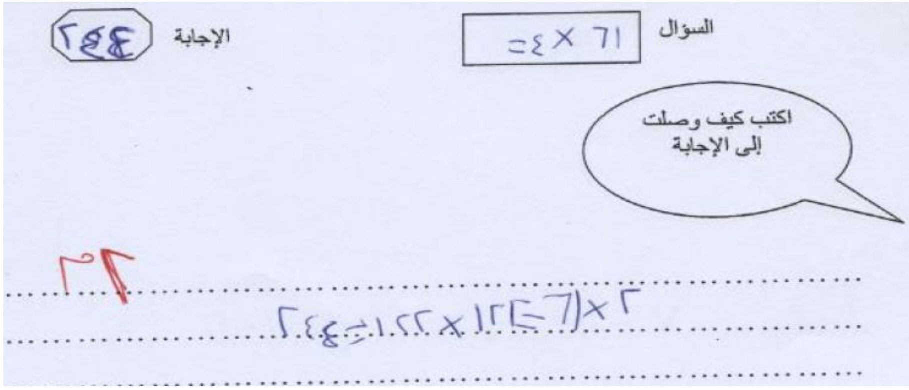


Figure 8: doubling strategy for multiplication task 61×4

In summary of the mental multiplication and division section, it seems to me that there is a lack of the use of mental strategies. Moreover, the data show that Saudi students make common errors when they do mental calculations and that they use standard algorithms as mental strategy, which leads to mistakes being made. In mental calculations, there is basic knowledge and skills should be considered by Saudi mathematics teachers in order to develop students' mental multiplication and division. For instance, the usefulness of the concepts of doubling and halving and the inverse relationship between them are important. Moreover, students need a lot of practices in multiplying by 10, 100, for example, to multiply by 15, multiply by 10, have the result, then add the result together. Using spreadsheets and computer programmes are helpful to practice mental calculation. However, in the following section, students' common errors when using standard algorithms of multiplication and division are discussed.

Standard algorithms of multiplication and division

The data of this investigation show there are different errors and a lack of understanding in terms of using standard algorithms to solve long multiplication and division tasks. There are common errors that are related either to the procedural use of algorithms or to the incomplete conceptual comprehension of such concepts, for example, place value.

Multiplication standard algorithms

Three multiplication tasks required students to use standard algorithms in the test; they included short algorithms TU×U (27×8), and long algorithms TU×TU (45×26), HTU×TU (318×29). As shown in table 8 that how the students' ability decreased from short algorithms to long algorithms regarding their ability to provide correct answers. The main reason for this decreased ability is that the number of mental and written calculations a student does to solve long algorithms influences the level of difficulty of such a task. This might consume a great deal of time in the teaching process and requires much practice by teachers and students. Therefore, some researchers, for example, Kamii & Dominick (1998), argue against spending much time in teaching long algorithms if students have a proper understanding of multiplication and division and have the ability to use calculators.

The analysis of students' incorrect multiplication answers shows the common error that students multiplied the two numbers of the problem, for example, 45×26 only vertically. They multiplied 5 by 6 and 4 by 2, then added the products together. This category of errors shows how some students might generalise standard algorithms of addition for multiplication. This is because of the similarity between addition and multiplication in terms of the standard vertical format that students tend to be taught to solve large numbers. Thus, teachers should be aware of this similarity when they teach the standard algorithms of multiplication for the first time.

Furthermore, a common error appears in long multiplication algorithms such as TU× TU. For example, 14% of incorrect answers of the task 45×26 were that students did not put zero before writing the answer when multiplying by 2 in 26 multiplied by 5. Students are taught to put zero because the value of 2 in 26 is 20 as shown in the next figure 9:

(The original picture) (The translated)

Figure 9: A common error based on lack of understanding the place value for the multiplication task 45×26.

This step or rule is taught to students because all the digits of the two numbers, which are multiplied by using standard algorithms, are treated as units. This error illustrates one of the problems of teaching standard algorithms based on the column value of numbers, for example, 52 is 5 tens and 2 units rather than quantity value 50 plus 2. Hansen et al. (2005) indicate that 'there is still concern about how teachers might facilitate children's understanding of standard algorithms which rely upon column value'.

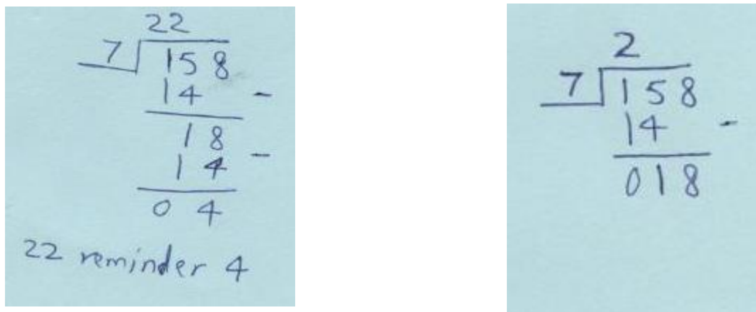
It is not surprising that incorrect multiplication facts caused the common errors in the two tasks 27×8 and 318×29. All numbers in the two tasks end in either 7, 8 or 9. For example, slightly more than a quarter of incorrect answers of 318×29 were due to the incorrect answer to the multiplication fact 8×9. This indicates the importance of the basic multiplication facts within the 1 to 10 times tables, not only as mathematical facts, but as a stepping stone for the development of further mathematical skills and knowledge. From my experience, teachers find that when teaching multiplication facts, it is difficult for them to make students learn these facts by heart. There are many suggested strategies in mathematics education research such as the 28 manageable facts (Hasselbring et al. (1988), Kilpatrick et al. (2001) cited in Wong and Evans (2007). However, teachers' awareness of the difficulty of teaching multiplication facts should encourage them to implement the strategies that have been discussed or suggested in research into classroom practices.

Division standard algorithms

As with standard algorithms of multiplication, half of the incorrect answers were due to a lack of understanding of using algorithms of division. In standard algorithms of long division, students follow the sequence of divide, multiply, subtract, and repeat this sequence until the end of the problem by 0 or the remainder. It seems to me that understanding each step of the sequence is very important when completing the problem. For example, in the tasks 672÷8 and 987÷47, there was a common difficulty in the first step of dividing 67 by 8 and 98 by 47. This first step requires a rigorous understanding of place value, multiplication facts, and an estimation of the answers of each division step.

One of the difficulties of long division algorithms was that some students were not able to complete algorithms by finding the remainder. For example; approximately half of the incorrect answers of the division sum 158÷7 were due to students being unable to complete the division. The following figure

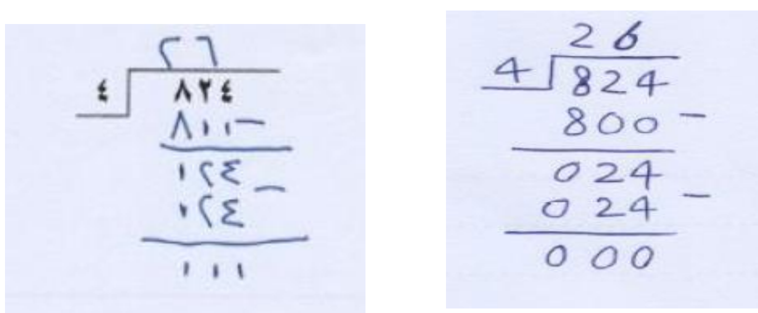
10 shows one correct solution and another uncompleted one because the answer included a remainder:



The correct algorithms Not completed answer

Figure 10: Comparison between correct algorithms and uncompleted answer for division with remainder. (The translated)

This difficulty of ending long division with a remainder indicates how more emphasis should be placed on the concept of a remainder for students to be able to make sense of it. Teaching and learning division with a remainder are essential for the teaching and learning of further concepts such as decimals and fractions. Another common error indicates the lack of understanding of the concept of place value and the reasonableness of an answer. The answer to the division task 824÷4 was 26 in 43% of the incorrect answers as shown by one student answer:



(The original) (The translated)

Figure 11: the common error in standard algorithms of the task 824÷ 4

Although this task 824÷4 can be solved mentally faster than when using standard algorithm, 13% of 96 Year Five students were able to solve it using algorithms. Consequently, it seems to me that solving mental calculations by various strategies plays a key role in comprehending standard algorithms. Thus, mental calculations should be taken into consideration by teachers when standard algorithms are introduced to students. Hopkins et al. (1999) suggest that there are two different possible approaches to introducing informal standard algorithms, as shown in the following table 10:

Analysing into small steps	**Analysing into substantial elements**
➢ Practise simple algorithms ➢ Introduce problems leading to simple algorithms.	➢ Develop children's mental maths skills.

➢ Practise harder algorithms ➢ Practise word problems leading to harder algorithms	➢ Introduce problems and encourage children to discuss the problems and develop their own methods. ➢ Work on estimation and a 'feel' for the size of the answer. ➢ Introduce standard methods, building on children's own effective non-standard methods. ➢ Practise problems leading to harder algorithms

Table 10: Two possible approaches to introducing standard algorithms by Hopkins et al. (1999, p.38)

The first approach, 'analysing into small steps', is the main one in the Saudi mathematics curriculum. However, it seems to me that the second approach is more appropriate than the first one, because standard algorithms include many mental calculations.

However, after a careful analysis of students' incorrect standard algorithms, I came to the conclusion that rote learning of the procedure of algorithms without students being able to make sense and have an understanding of the concept might be the main factor that causes the errors discussed above. Thus, there is a serious need to rethink the teaching strategies of standard algorithms for multiplication and division. Hopkins et al. (1999, p.30) indicate that 'when children need to use pencil and paper methods because the numbers are too difficult to hold in the head, they can be introduced to standard algorithms, based on confident mental methods, numerical knowledge and a secure understanding of place value'.

In order to improve the performance of students in standard algorithms, teachers are encouraged to start by informal written methods such as grid method and partitioning for multiplication algorithms and estimating by using multiples of the divisor for division algorithms. For example, the long multiplication 42×65 and division 1730÷48 can be taught by informal methods before introducing standard algorithms as the following figure shows:

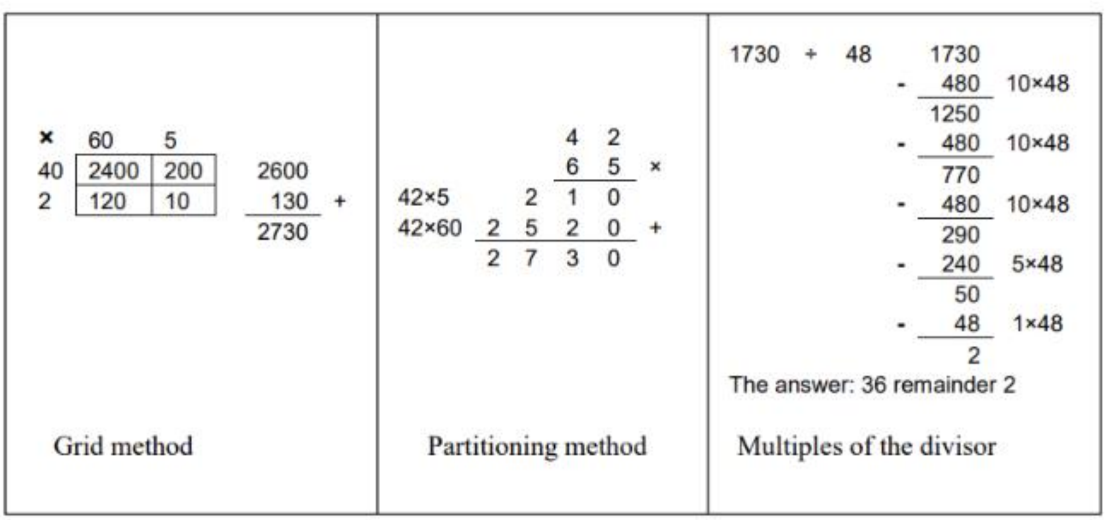


Figure 12: Informal written methods of the long multiplication 42×65 and division 1730÷48.

The main idea of informal written methods is that to teach students different alternative methods with preparing students for standard algorithms. One of my criticisms to Saudi mathematics curriculum and teaching strategies is that following only one way to teach standard algorithms and without giving students opportunities to try their own methods. Steffe (1994, p 8) say that 'it is a drastic mistake to ignore child-generated algorithms in favour of the "standard" paper and pencil algorithms'.

CONCLUSION

This study has investigated the misconceptions and common errors of multiplication and division of whole numbers that exist among 96 grade five Saudi students. It has covered students' understanding of the concepts and properties of multiplication and division. Additionally, the common errors related to mental and standard algorithms were discussed in this investigation. The study used qualitative and quantitative methods to collect the data. All 96 students were required to complete a standardised test and 10 of the students were chosen to be interviewed. The findings indicate that some Saudi 12 year olds have initial misunderstandings about the concepts of multiplication and division, which leads to certain misconceptions such as "multiplication always make bigger" and "division always make smaller". One common error among students is the solution to tasks multiplying and dividing multipliers of 10. Moreover, students make common mistakes and have misunderstandings about mental calculations and standard algorithms, such as using incorrect procedures for algorithms. Although mental calculations are critical skills for learning mathematics, the study shows that there is lack of the use of mental strategies by Saudi students for multiplication and division.

One of the most important aspects of this study is that the findings might support the development of mathematics teachers' pedagogical knowledge of teaching multiplication and division of whole numbers in high grades at primary level. Furthermore, understanding the implications of identifying and addressing students' common errors and misconceptions is essential for the teaching and learning of mathematics in terms of improving the mathematics curriculum, the role of teachers, and teaching strategies in order to reduce such errors and misconceptions.

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